# 9 Optimization

## 9.1 - Matrix Games

**1.** $$\begin{array}{cc} & \begin{array}{cc} d & q \end{array} \\ \begin{array}{c} d \\ q \end{array} & \begin{bmatrix} -10 & 10 \\ 25 & -25 \end{bmatrix} \end{array}$$

**2.** $$\begin{array}{cc} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 2 & -3 & 4 \\ -3 & 4 & -5 \\ 4 & -5 & 6 \end{bmatrix} \end{array}$$

**3.** $$\begin{array}{cc} & \begin{array}{ccc} r & s & p \end{array} \\ \begin{array}{c} \text{rock} \\ \text{scissors} \\ \text{paper} \end{array} & \begin{bmatrix} 0 & 5 & -5 \\ -5 & 0 & 5 \\ 5 & -5 & 0 \end{bmatrix} \end{array}$$

**4.** $$\begin{array}{cc} & \begin{array}{cc} r4 & b9 \end{array} \\ \begin{array}{c} r3 \\ r6 \\ b7 \end{array} & \begin{bmatrix} 4 & -12 \\ 6 & -15 \\ -11 & 9 \end{bmatrix} \end{array}$$

**5.** $\begin{bmatrix} 4 & 3 \\ 1 & -1 \end{bmatrix}$ **6.** $\begin{bmatrix} 2 & 1 & 3 \\ 4 & -2 & 1 \end{bmatrix}$ **7.** $\begin{bmatrix} 5 & 3 & 4 & 3 \\ -2 & 1 & -5 & 2 \\ 4 & 3 & 7 & 3 \end{bmatrix}$ **8.** $\begin{bmatrix} -2 & 4 & 1 & -1 \\ 3 & 5 & 2 & 2 \\ 1 & -3 & 0 & 2 \end{bmatrix}$

**9. a.** $E(\mathbf{x},\mathbf{y}) = \frac{13}{12},\ v(\mathbf{x}) = \min\left\{\frac{5}{6}, 1, \frac{9}{6}\right\} = \frac{5}{6},\ v(\mathbf{y}) = \max\left\{\frac{3}{4}, \frac{3}{2}, \frac{1}{2}\right\} = \frac{3}{2}$

**b.** $E(\mathbf{x},\mathbf{y}) = \frac{9}{8},\ v(\mathbf{x}) = \min\left\{1, \frac{3}{4}, \frac{7}{4}\right\} = \frac{3}{4},\ v(\mathbf{y}) = \max\left\{\frac{1}{2}, \frac{5}{4}, \frac{3}{2}\right\} = \frac{3}{2}$

**10. a.** $E(\mathbf{x},\mathbf{y}) = -\frac{1}{4},\ v(\mathbf{x}) = \min\left\{\frac{4}{3}, -\frac{4}{3}, \frac{5}{3}, \frac{1}{3}\right\} = -\frac{4}{3},\ v(\mathbf{y}) = \max\left\{\frac{1}{4}, \frac{1}{4}, -\frac{1}{2}\right\} = \frac{1}{4}$

**b.** $E(\mathbf{x},\mathbf{y}) = \frac{1}{8},\ v(\mathbf{x}) = \min\left\{1, -\frac{1}{4}, \frac{1}{2}, -\frac{1}{4}\right\} = -\frac{1}{4},\ v(\mathbf{y}) = \max\left\{\frac{1}{4}, -\frac{3}{4}, \frac{3}{4}\right\} = \frac{3}{4}$

**11.** Given $A = \begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix}$, graph $\begin{cases} z = 3(1-t)+(0)t = 3-3t \\ z = -2(1-t)+(1)t = -2+3t \end{cases}$.

The lines intersect at $(t, z) = \left(\frac{5}{6}, \frac{1}{2}\right)$. The optimal row strategy is $\hat{\mathbf{x}} = \mathbf{x}\left(\frac{5}{6}\right) = \begin{bmatrix} 1-\frac{5}{6} \\ \frac{5}{6} \end{bmatrix} = \begin{bmatrix} \frac{1}{6} \\ \frac{5}{6} \end{bmatrix}$, and the value of the game is $v = \frac{1}{2}$. By Theorem 4, the optimal column strategy $\hat{\mathbf{y}}$ satisfies $E(\mathbf{e}_1, \hat{\mathbf{y}}) = \frac{1}{2}$ and $E(\mathbf{e}_2, \hat{\mathbf{y}}) = \frac{1}{2}$ because $\hat{\mathbf{x}}$ is a linear combination of both $\mathbf{e}_1$ and $\mathbf{e}_2$. From the second of these conditions, $\frac{1}{2} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c_2$. From this, $c_1 = \frac{1}{2}$ and $\hat{\mathbf{y}} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$. As a check on this work, one can compute $E(\mathbf{e}_1, \hat{\mathbf{y}}) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 3 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \frac{1}{2}$.

**12.** $\hat{\mathbf{x}} = \begin{bmatrix} \frac{9}{13} \\ \frac{4}{13} \end{bmatrix}$, $\hat{\mathbf{y}} = \begin{bmatrix} \frac{8}{13} \\ \frac{5}{13} \end{bmatrix}$, $v = \frac{6}{13}$

**13.** Given $A = \begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix}$, graph $\begin{cases} z = 3(1-t) + (4)t = 3 + t \\ z = 5(1-t) + (1)t = 5 - 4t \end{cases}$.

The lines intersect at $(t, z) = (\frac{2}{5}, \frac{17}{5})$. The optimal row strategy is $\hat{\mathbf{x}} = \mathbf{x}(\frac{2}{5}) = \begin{bmatrix} 1 - \frac{2}{5} \\ \frac{2}{5} \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{2}{5} \end{bmatrix}$, and the value of the game is $v = \frac{17}{5}$. By Theorem 4, the optimal column strategy $\hat{\mathbf{y}}$ satisfies $E(\mathbf{e}_1, \hat{\mathbf{y}}) = \frac{17}{5}$ and $E(\mathbf{e}_2, \hat{\mathbf{y}}) = \frac{17}{5}$ because $\hat{\mathbf{x}}$ is a linear combination of both $\mathbf{e}_1$ and $\mathbf{e}_2$. From the first of these conditions,

$$\frac{17}{5} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ 1 - c_1 \end{bmatrix} = \begin{bmatrix} 3 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ 1 - c_1 \end{bmatrix} = 5 - 2c_1$$

From this, $c_1 = \frac{4}{5}$ and $\hat{\mathbf{y}} = \begin{bmatrix} \frac{4}{5} \\ \frac{1}{5} \end{bmatrix}$. As a check on this work, one can compute

$$E(\mathbf{e}_2, \hat{\mathbf{y}}) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} \frac{4}{5} \\ \frac{1}{5} \end{bmatrix} = \begin{bmatrix} 4 & 1 \end{bmatrix} \begin{bmatrix} \frac{4}{5} \\ \frac{1}{5} \end{bmatrix} = \frac{17}{5}$$

**14.** Columns 2 and 3 dominate column 1, so the column player will never choose column 2 or column 3. The new game is $\begin{bmatrix} 3 & * & * & 2 \\ -1 & * & * & 8 \end{bmatrix}$. Let $B = \begin{bmatrix} 3 & 2 \\ -1 & 8 \end{bmatrix}$. Graph

$$\begin{cases} z = 3(1-t) + (-1)t = -4t + 3 \\ z = 2(1-t) + \quad 8t = 6t + 2 \end{cases}$$

Solve for the intersection, to get $t = .1$, and $\hat{\mathbf{x}} = \mathbf{x}(.1) = \begin{bmatrix} 1 - .1 \\ .1 \end{bmatrix} = \begin{bmatrix} .9 \\ .1 \end{bmatrix}$. The game value is 6(.1) + 2 = 2.6. Let $\mathbf{y} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, and set 2.6 = $E(\mathbf{e}_1, \mathbf{y}) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ -1 & 8 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, so $3c_1 + 2c_2 = 2.6$. Since $\mathbf{y}$ is a probability vector, $3c_1 + 2(1 - c_1) = 2.6$, and $c_1 = .6$. Thus, $c_2 = 1 - .6 = .4$, and the optimal column strategy $\mathbf{y}$ for the matrix game $B$ has entries .6 and .4. The optimal $\hat{\mathbf{y}}$ for the matrix game $A$ has four entries.

The game matrix, written as $\begin{bmatrix} 3 & * & * & 2 \\ -1 & * & * & 8 \end{bmatrix}$, shows that $\hat{\mathbf{y}} = \begin{bmatrix} .6 \\ 0 \\ 0 \\ .4 \end{bmatrix}$ and, from above, $\hat{\mathbf{x}} = \begin{bmatrix} .9 \\ .1 \end{bmatrix}$.

**15.** Column 2 dominates column 3, so the column player $C$ will never play column 2. The graph shows why column 2 will not affect the column play, and the graph shows that the value of the game is 2.

The new game is $\begin{bmatrix} 4 & * & 2 & 0 \\ 1 & * & 2 & 5 \end{bmatrix}$. Let $B = \begin{bmatrix} 4 & 2 & 0 \\ 1 & 2 & 5 \end{bmatrix}$. The line for column 3 is $z = 2$. That line intersects the line for column 4 where $z = 0(1 - t) + 5t = 2$, and $t = .4$. An optimal row strategy is $\hat{\mathbf{x}} = \begin{bmatrix} 1-.4 \\ .4 \end{bmatrix} = \begin{bmatrix} .6 \\ .4 \end{bmatrix}$. Another optimal row strategy is determined by the intersection of the lines for columns 1 and 3, where $z = 4(1 - t) + t = 2$, $t = \frac{2}{3}$, and $\hat{\mathbf{x}} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$. It can be shown that any convex combination of these two optimal strategies is also an optimal row strategy.

To find the optimal column strategy, set $\mathbf{y} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$, and set $2 = E(\mathbf{e}_1, \mathbf{y}) = \mathbf{e}_1{}^T B\mathbf{y}$ and $2 = E(\mathbf{e}_2\ \mathbf{y}) = \mathbf{e}_2{}^T B\mathbf{y}$. These two equations produce $4c_1 + 2c_2 = 2$ and $c_1 + 2c_2 + 5c_3 = 2$. Combine these with the fact that $c_1 + c_2 + c_3$ must be 1, and solve the system:

$$\begin{bmatrix} 4 & 2 & 0 & 2 \\ 1 & 2 & 5 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \; c_2 = 1, \text{ and } \mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

This is the column strategy for the game matrix $B$. For $A$, $\hat{\mathbf{y}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

**16.** Row 3 is recessive to row 2, so the row player $R$ will never play row 3. Also, column 3 dominates column 2, so the column player $C$ will never play column 3. Thus, the game reduces:

$$A = \begin{bmatrix} 5 & -1 & 1 \\ 4 & 2 & 1 \\ -2 & -3 & 1 \end{bmatrix} \to \begin{bmatrix} 5 & -1 & 1 \\ 4 & 2 & 2 \\ * & * & * \end{bmatrix} \to \begin{bmatrix} 5 & -1 & * \\ 4 & 2 & * \\ * & * & * \end{bmatrix}$$

Let $B = \begin{bmatrix} 5 & -1 \\ 4 & 2 \end{bmatrix}$. The row minima are $-1$ and 2, so the max of the minima is 2. The column maxima are 5 and 2, so the min of the maxima is 2. Thus, the value of the game is 2, and game $B$ has a saddle point, where $R$ always plays row 2 and $C$ always plays column 2. For the original game, the optimal solutions are $\hat{\mathbf{x}} = \hat{\mathbf{y}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Another solution method is to check the original matrix for a saddle point and find it directly, without reducing the size of the matrix.

**17.** Row 2 is recessive to row 3, and row 4 is recessive to row 1, so the row player $R$ will never play row 2 or row 4. Also, column 4 dominates column 2, so the column player $C$ will never play column 4. Thus, the game reduces:

$$A=\begin{bmatrix}0 & 1 & -1 & 4 & 3\\ 1 & -1 & 3 & -1 & -3\\ 2 & -1 & 4 & 0 & -2\\ -1 & 0 & -2 & 2 & 1\end{bmatrix}\to\begin{bmatrix}0 & 1 & -1 & 4 & 3\\ * & * & * & * & *\\ 2 & -1 & 4 & 0 & -2\\ * & * & * & * & *\end{bmatrix}\to\begin{bmatrix}0 & 1 & -1 & * & 3\\ * & * & * & * & *\\ 2 & -1 & 4 & * & -2\\ * & * & * & * & *\end{bmatrix}$$

Let $B=\begin{bmatrix}0 & 1 & -1 & 3\\ 2 & -1 & 4 & -2\end{bmatrix}$. (If column 4 in $A$ is not noticed as dominant, this fact will become clear after the lines are plotted for the columns of the reduced matrix.) The equations of the lines corresponding to the columns of $B$ are

(column 1) $z = 0(1-t)+2t \;= 2t$
(column 2) $z = 1(1-t)-t \;= 1-2t$
(column 3) $z = -1(1-t)+4t = -1+5t$
(column 4) $z = 3(1-t)-2t \;= 3-5t$

The graph of $v(\mathbf{x}(t))$ as a function of $t$ is the polygonal path formed by line 3 (for column 3), then line 1 (column 1), and then line 2 (column 2). The highest point on this path occurs at the intersection of lines 3 and 2. Solve $z=-1+5t$ and $z=1-2t$ to find $t=\frac{2}{7}$ and $z=\frac{3}{7}$. The value of game $B$ is $z=\frac{3}{7}$, attained when $\hat{\mathbf{x}}=\begin{bmatrix}1-\frac{2}{7}\\ \frac{2}{7}\end{bmatrix}=\begin{bmatrix}\frac{5}{7}\\ \frac{2}{7}\end{bmatrix}$.

Because columns 2 and 3 of $B$ determine the optimal solution, the optimal strategy for the column player $C$ is a convex combination $\hat{\mathbf{y}}$ of the pure column strategies $\mathbf{e}_2$ and $\mathbf{e}_3$, say, $\hat{\mathbf{y}}=\begin{bmatrix}0\\ c_2\\ c_3\\ 0\end{bmatrix}$. Since both coordinates of the optimal row solution are nonzero, Theorem 4 shows that $E(\mathbf{e}_i,\hat{\mathbf{y}})=\frac{3}{7}$ for $i = 1, 2$. Each condition, by itself, determines $\hat{\mathbf{y}}$. For example,

$$E(\mathbf{e}_1,\hat{\mathbf{y}})=\mathbf{e}_1^TB\hat{\mathbf{y}}=\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}0 & 1 & -1 & 3\\ 2 & -1 & 4 & -2\end{bmatrix}\begin{bmatrix}0\\ c_2\\ c_3\\ 0\end{bmatrix}=c_2-c_3=\frac{3}{7}$$

Substitute $c_3 = 1 - c_2$, and obtain $c_2=\frac{5}{7}$ and $c_3=\frac{2}{7}$. Thus, $\hat{\mathbf{y}}=\begin{bmatrix}0\\ \frac{5}{7}\\ \frac{2}{7}\\ 0\end{bmatrix}$ is the optimal column strategy for game $B$. For game $A$, $\hat{\mathbf{x}}=\begin{bmatrix}\frac{5}{7}\\ 0\\ \frac{2}{7}\\ 0\end{bmatrix}$ and $\hat{\mathbf{y}}=\begin{bmatrix}0\\ \frac{5}{7}\\ \frac{2}{7}\\ 0\\ 0\end{bmatrix}$, and the value of the game is still $\frac{3}{7}$.

**18.** Row 2 is recessive to row 4, and row 3 is recessive to row 1, so the row player $R$ will never play row 2 or row 3. After these rows are removed, column 4 dominates column 2, so the column player $C$ will never play column 4. Thus, the game reduces:

$$A=\begin{bmatrix} 6 & 4 & 5 & 5 \\ 0 & 4 & 2 & 7 \\ 6 & 3 & 5 & 2 \\ 2 & 5 & 3 & 7 \end{bmatrix} \to \begin{bmatrix} 6 & 4 & 5 & 5 \\ * & * & * & * \\ * & * & * & * \\ 2 & 5 & 3 & 7 \end{bmatrix} \to \begin{bmatrix} 6 & 4 & 5 & * \\ * & * & * & * \\ * & * & * & * \\ 2 & 5 & 3 & * \end{bmatrix}$$

Let $B=\begin{bmatrix} 6 & 4 & 5 \\ 2 & 5 & 3 \end{bmatrix}$. (If column 4 in $A$ is not noticed as dominant, this fact will become clear after the lines are plotted for the columns of the reduced matrix.) The equations of the lines corresponding to the columns of $B$ are

(column 1) $z = 6(1-t) + 2t = 6 - 4t$
(column 2) $z = 4(1-t) + 5t = 4 + t$
(column 3) $z = 5(1-t) + 3t = 5 - 2t$

The graph of $v(\mathbf{x}(t))$ as a function of $t$ is the polygonal path formed by line 2 (for column 2), then line 3 (column 3), and then line 1 (column 1). The highest point on this path occurs at the intersection of lines 2 and 3. Solve $z = 4 + t$ and $z = 5 - 2t$ to find $t = \frac{1}{3}$ and $z = \frac{13}{3}$. The value of game $B$ is $z = \frac{13}{3}$, attained when $\hat{\mathbf{x}}=\begin{bmatrix} 1-\frac{1}{3} \\ \frac{1}{3} \end{bmatrix}=\begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$. Because columns 2 and 3 of $B$ determine the optimal solution, the optimal strategy for the column player $C$ is a convex combination $\hat{\mathbf{y}}$ of the pure column strategies $\mathbf{e}_2$ and $\mathbf{e}_3$, say, $\hat{\mathbf{y}}=\begin{bmatrix} 0 \\ c_2 \\ c_3 \end{bmatrix}$. Since both coordinates of the optimal row solution are nonzero, Theorem 4 shows that $E(\mathbf{e}_i, \hat{\mathbf{y}})=\frac{13}{3}$ for $i = 1, 2$. Each condition, by itself, determines $\hat{\mathbf{y}}$. For example,

$$\tfrac{13}{3}=E(\mathbf{e}_1,\hat{\mathbf{y}})=\mathbf{e}_1^T B\hat{\mathbf{y}}=\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 6 & 4 & 5 \\ 2 & 5 & 3 \end{bmatrix}\begin{bmatrix} 0 \\ c_2 \\ c_3 \end{bmatrix}=4c_2+5c_3=4c_2+5(1-c_2)=5-c_2$$

Then $c_2 = \frac{2}{3}$ and $c_3 = \frac{1}{3}$. Thus, $\hat{\mathbf{y}}=\begin{bmatrix} 0 \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$ is the optimal column strategy for game $B$. For game $A$,

$\hat{\mathbf{x}}=\begin{bmatrix} \frac{2}{3} \\ 0 \\ 0 \\ \frac{1}{3} \end{bmatrix}$ and $\hat{\mathbf{y}}=\begin{bmatrix} 0 \\ \frac{2}{3} \\ \frac{1}{3} \\ 0 \end{bmatrix}$, and the value of the game is still $\frac{13}{3}$.

**19. a.** Army: 1/3 river, 2/3 land; guerrillas: 1/3 river, 2/3 land; 2/3 of the supplies get through.

**b.** Army: 7/11 river, 4/11 land; guerrillas: 7/11 river, 4/11 land; 64/121 of the supplies get through.

**20. a.** Army: 7/11 river, 4/11 land; guerrillas: 9/11 river, 2/11 land.

**b.** The value of the game is –36/11. This means the army will average 36/11 casualties a day.

**21.** True. See the paragraph above Example 1.

**22.** True. See the definition of a saddle point.

**23.** True. With a pure strategy, a player chooses one particular play with probability 1.

**24.** False. A strategy is optimal only if its value equals the value of the game.

**25.** False. $v(\mathbf{x})$ is equal to the *minimum* of the inner product of **x** with each of the columns of the payoff matrix.

**26.** True. See the definition of $v_R$.

**27.** False. The Minimax Theorem says only that the value of a game is the same for both players. It does not guarantee that there is an optimal mixed strategy for each player that produces this common value. It is the Fundamental Theorem for Matrix Games that says every matrix game has a solution.

**28.** False. It guarantees the existence of a solution, but it does not show how to find a solution.

**29.** True. By Theorem 5, row *r* may be deleted from the payoff matrix, and any optimal strategy from the new matrix will also be an optimal strategy for matrix *A*. This optimal strategy will not involve row *s*.

**30.** True. By Theorem 5, the dominating column *t* may be deleted from the payoff matrix, and any optimal strategy from the new matrix will also be an optimal strategy for matrix *A*. This optimal strategy will not involve column *t*. (Note, however, that if a *column* is *recessive*, it may or may not be nonzero in an optimal mixed strategy. In Example 6, column 4 is recessive to column 1, but column 4 has probability 0 in the optimal mixed strategy for *C*. However, column 3 is also recessive to column 1, and the probability of column 3 in the optimal strategy is positive.)

**31.** $\hat{\mathbf{x}} = \begin{bmatrix} \frac{1}{6} \\ \frac{5}{6} \\ 0 \end{bmatrix}, \; \hat{\mathbf{y}} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \; v = 0$

**32. a.** $\begin{bmatrix} 1 & -3 \\ -2 & 4 \end{bmatrix}$ **b.** $-A^T$

**33. a.** To find $\hat{\mathbf{x}}$ when $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, let $\mathbf{x}(t) = \begin{bmatrix} 1-t \\ t \end{bmatrix}$, where $0 \le t \le 1$. Then, as in Example 4, solve the system of linear equations $\begin{matrix} z = a(1-t) + ct \\ z = b(1-t) + dt \end{matrix}$ to see that $t = \dfrac{a-b}{a-b+d-c}$ and $z = \dfrac{ad-bc}{a-b+d-c}$.

Note that $a - b + d - c \neq 0$, for if it were 0 then the graphs of the two linear equations would be parallel (having the same slope) and matrix *A* would have a saddle point. Since *A* has no saddle point, the two graphs must intersect at $(t, z)$ and *t* must satisfy $0 \le t \le 1$. We compute

$1 - t = 1 - \dfrac{a-b}{a-b+d-c} = \dfrac{d-c}{a-b+d-c}$ and obtain $\hat{\mathbf{x}} = \begin{bmatrix} \dfrac{d-c}{a-b+d-c} \\ \dfrac{a-b}{a-b+d-c} \end{bmatrix}$. We have $v = z$, and we use Theorem 4 to find $\hat{\mathbf{y}}$ as in Example 5.

**b.** To show that the optimal strategies for $A$ and $\alpha A+\beta J$ are the same, note that $\alpha A+\beta J=\begin{bmatrix}\alpha a+\beta & \alpha b+\beta\\ \alpha c+\beta & \alpha d+\beta\end{bmatrix}$. Applying the formula for $\hat{\mathbf{x}}$ to the matrix $\alpha A+\beta J$ we obtain

$$\hat{\mathbf{x}}=\begin{bmatrix}\dfrac{(\alpha d+\beta)-(\alpha c+\beta)}{(\alpha a+\beta)-(\alpha b+\beta)+(\alpha d+\beta)-(\alpha c+\beta)}\\ \dfrac{(\alpha a+\beta)-(\alpha b+\beta)}{(\alpha a+\beta)-(\alpha b+\beta)+(\alpha d+\beta)-(\alpha c+\beta)}\end{bmatrix}=\begin{bmatrix}\dfrac{\alpha d-\alpha c}{\alpha a-\alpha b+\alpha d-\alpha c}\\ \dfrac{\alpha a-\alpha b}{\alpha a-\alpha b+\alpha d-\alpha c}\end{bmatrix}=\begin{bmatrix}\dfrac{d-c}{a-b+d-c}\\ \dfrac{a-b}{a-b+d-c}\end{bmatrix}.$$

But this is the same as $\hat{\mathbf{x}}$ for $A$. A similar argument applies to $\hat{\mathbf{y}}$.

**34.** Let $A=\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}$, $\mathbf{x}=\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}$, and $\mathbf{y}=\begin{bmatrix}1\\ 0\end{bmatrix}$. Then $v = 1$ and $E(\mathbf{x}, \mathbf{y}) = 1$, but $\mathbf{y}$ is not optimal. There are many other possibilities.

# 9.2 - Linear Programming – Geometric Method

**1.** Let $x_1$ be the amount invested in mutual funds, $x_2$ the amount in CDs, and $x_3$ the amount in savings. Then $\mathbf{b}=\begin{bmatrix}12{,}000\\ 0\\ 0\end{bmatrix}$, $\mathbf{x}=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$, $\mathbf{c}=\begin{bmatrix}.11\\ .08\\ .06\end{bmatrix}$, and $A=\begin{bmatrix}1 & 1 & 1\\ 1 & -1 & -1\\ 0 & 1 & -2\end{bmatrix}$.

**2.** Let $x_1$ be the number of bags of Pixie Power, and $x_2$ the number of bags of Misty Might. Then

$$\mathbf{b}=\begin{bmatrix}28\\ 30\\ 20\\ 25\end{bmatrix},\ \mathbf{x}=\begin{bmatrix}x_1\\ x_2\end{bmatrix},\ \mathbf{c}=\begin{bmatrix}50\\ 40\end{bmatrix},\text{ and } A=\begin{bmatrix}3 & 2\\ 2 & 4\\ 1 & 3\\ 2 & 1\end{bmatrix}.$$

**3.** $\mathbf{b}=\begin{bmatrix}20\\ -10\end{bmatrix}$, $\mathbf{c}=\begin{bmatrix}3\\ 4\\ -2\end{bmatrix}$, $A=\begin{bmatrix}1 & 2 & 0\\ 0 & 3 & -5\end{bmatrix}$

**4.** $\mathbf{b}=\begin{bmatrix}25\\ 40\\ -40\end{bmatrix}$, $\mathbf{c}=\begin{bmatrix}3\\ 1\\ 5\end{bmatrix}$, $A=\begin{bmatrix}5 & 7 & 1\\ 2 & 3 & 4\\ -2 & -3 & -4\end{bmatrix}$

**5.** $\mathbf{b}=\begin{bmatrix}-35\\ 20\\ -20\end{bmatrix}$, $\mathbf{c}=\begin{bmatrix}-7\\ 3\\ -1\end{bmatrix}$, $A=\begin{bmatrix}-1 & 4 & 0\\ 0 & 1 & -2\\ 0 & -1 & 2\end{bmatrix}$

**6.** $\mathbf{b}=\begin{bmatrix}27\\ -40\end{bmatrix}$, $\mathbf{c}=\begin{bmatrix}-1\\ -5\\ 2\end{bmatrix}$, $A=\begin{bmatrix}2 & 1 & 4\\ -1 & 6 & -3\end{bmatrix}$

**7.** First, find the intersection points for the bounding lines:

(1) $2x_1 + x_2 = 32$, (2) $x_1 + x_2 = 18$, (3) $x_1 + 3x_2 = 24$

Even a rough sketch of the graphs of these lines will reveal that (0, 0), (16, 0), and (0, 8) are vertices of the feasible set. What about the intersections of the lines corresponding to (1), (2), and (3)?

The graphical method will work, provided the graph is large enough and is drawn carefully. In many simple problems, even a small sketch will reveal which intersection points are vertices of the feasible set. In this problem, however, three intersection points happen to be quite close to each other, and a slight inaccuracy on a graph of size 3"×3" or smaller may lead to an incorrect solution. In a case such as this, the following algebraic procedure will work well:

When an intersection point is found that corresponds to two inequalities, test it in the other inequalities to see whether the point is in the feasible set.

The intersection of (1) and (2) is (14, 4). Test this in the third inequality: (14) + 3(4) = 26 > 24. The intersection point does not satisfy the inequality for (3), so (14, 4) is **not** in the feasible set.

The intersection of (1) and (3) is (14.4, 3.2). Test this in the second inequality: 14.4 + 3.2 = 17.6 ≤ 18, so (14.4, 3.2) **is** in the feasible set.

The intersection of (2) and (3) is (15, 3). Test this in the first inequality: 2(15) + (3) = 33 > 32, so (15, 3) is **not** in the feasible set.

Next, list the vertices of the feasible set: (0, 0), (16, 0), (14.4, 3.2), and (0, 8). Then compute the values of the objective function $80x_1 + 65x_2$ at these points.

| | |
|---|---|
| (0, 0): | 80(0) + 65(0) = 0 |
| (16, 0): | 80(16) + 3(0) = 1280 |
| (14.4, 3.2): | 80(14.4) + 65(3.2) = 1360 |
| (0, 8): | 80(0) + 65(8) = 520 |

Finally, select the maximum of the objective function, which is 1360, and note that this maximum is attained at (14.4, 3.2).

**8.** First, convert the problem to a canonical (maximization) problem:

Maximize $-5x_1 - 3x_2$, subject to

(1) $-2x_1 - 5x_2 \le -10$, (2) $-3x_1 - x_2 \le -6$, (3) $-x_1 - 7x_2 \le -7$

Next, find the intersection points for the bounding lines. The intersection of the equalities for (1) and (2) is $(\frac{20}{13}, \frac{18}{13})$. Test this in the inequality (3): $-(\frac{20}{13}) - 7(\frac{18}{13}) = -\frac{146}{13} < -7$. This point satisfies (3), so $(\frac{20}{13}, \frac{18}{13})$ **is** in the feasible set.

The intersection corresponding to (1) and (3) is $(\frac{35}{9}, \frac{4}{9})$. Test this in (2): $-3(\frac{35}{9}) - (\frac{4}{9}) = -\frac{109}{9} < -6$, so $(\frac{35}{9}, \frac{4}{9})$ **is** in the feasible set.

The intersection corresponding to (2) and (3) is $(\frac{7}{4}, \frac{3}{4})$. Test this in (1): $-2(\frac{7}{4}) - 5(\frac{3}{4}) = -\frac{29}{4} > -10$, so $(\frac{7}{4}, \frac{3}{4})$ is **not** in the feasible set.

The vertices of the feasible set are (0, 6), $(\frac{20}{13}, \frac{18}{13})$, $(\frac{35}{9}, \frac{4}{9})$, and (7, 0). The values of the objective function $-5x_1 - 3x_2$ at these points are $-18$, $-\frac{154}{13} \approx -11.85$, $-\frac{187}{9} \approx -20.8$, and $-35$, respectively. The maximum value of the objective function $-5x_1 - 3x_2$ is $-\frac{154}{13}$, which occurs at $(\frac{20}{13}, \frac{18}{13})$. So the *minimum* value of the *original* objective function $5x_1 + 3x_2$ is $\frac{154}{13}$, and this occurs at $(\frac{20}{13}, \frac{18}{13})$.

**9.** Unbounded.

**10.** Infeasible.

**11.** True. See the definition of a canonical linear programming problem.

**12.** True. This is a logically equivalent version (called the *contrapositive*) of Theorem 6.

**13.** False. The vector $\bar{\mathbf{x}}$ must itself be feasible. It is possible for a nonfeasible vector (as well as the optimal solution) to yield the maximum value of $f$.

**14.** False. Theorem 6 says that some extreme point is an optimal solution, but not every optimal solution must be an extreme point.

**15.** First, find the intersection points for the bounding lines:

(1) $3x_1 + 2x_2 = 1200$ (fescue), (2) $x_1 + 2x_2 = 800$ (rye), (3) $x_1 + x_2 = 450$ (bluegrass)

The intersection of lines (1) and (2) is (200, 300). Test this in the inequality corresponding to (3): (200) + (300) = 500 > 450. The intersection point does not satisfy the inequality for (3), so (200, 300) is **not** in the feasible set.

The intersection of (1) and (3) is (300, 150). Test this in (2): (300) + 2(150) = 600 < 800, so (300, 150) **is** in the feasible set.

The intersection of (2) and (3) is (100, 350). Test this in (1): 3(100) + 2(350) = 1000 < 1200, so (100, 350) **is** in the feasible set.

The vertices of the feasible set are (0, 0), (400, 0), (300, 150), (100, 350), and (0, 400). Evaluate the objective function at each vertex:

| | |
|---|---|
| (0, 0): | 2(0) + 3(0) = 0 |
| (400, 0): | 2(400) + 3(0) = 800 |
| (300, 150): | 2(300) + 3(150) = 1050 |
| (100, 350): | 2(100) + 3(350) = 1250 |
| (0, 400): | 2(0) + 3(400) = 800 |

The maximum of the objective function $2x_1 + 3x_2$ is \$1250 at (100, 350).

**16.** First, find the intersection points for the bounding lines:

(1) $12x_1 + 4x_2 = 48$, (2) $4x_1 + 4x_2 = 32$, (3) $x_1 + 5x_2 = 20$

The intersection of lines (1) and (2) is (2, 6). Test this in the third inequality: (2) + 5(6) = 32 > 20. The intersection point satisfies the inequality for (3), so (2, 6) **is** in the feasible set.

The intersection of (1) and (3) is (192/56, 160/56). Test this in the second inequality: 4(192/56) + 4(160/56) = 1408/56 = 176/7 ≈ 25.16 < 32, so this point is **not** in the feasible set.

The intersection of (2) and (3) is (5, 3). Test this in the first inequality: 12(5) + 4(3) = 72 > 48, so (5, 3) **is** in the feasible set.

The vertices of the feasible set are (20, 0), (5, 3), (2, 6), and (0, 12). Evaluate the objective function at each vertex. (The values here represent thousands of dollars.)

(20, 0): 3.5(20) + 3(0) = 70

(5, 3): 3.5(5) + 3(3) = 26.5

(2, 6): 3.5(2) + 3(6) = 25

(0, 12): 3.5(0) + 3(12) = 36

The minimum cost is $25,000, when the production schedule is $(x_1, x_2) = (2, 6)$. That is, the cost is minimized when refinery A runs for 2 days and refinery B runs for 6 days.

**17.** First, find the intersection points for the bounding lines:

(1) $5x_1 + 2x_2 = 200$, (2) $.2x_1 + .4x_2 = 16$, (3) $.2x_1 + .2x_2 = 10$

The intersection of (1) and (2) is (30, 25). Test this in the third inequality: .2(30) + .2(25) = 11 > 10. The intersection point does not satisfy the inequality for (3), so (30, 25) is **not** in the feasible set.

The intersection of (1) and (3) is (100/3, 100/6). Test this in the second inequality: 2(100/3) + .4(100/6) = 13.3 < 16, so (100/3, 100/6) **is** in the feasible set.

The intersection of (2) and (3) is (20, 30). Test this in the first inequality: 5(20) + 2(30) = 160 < 200, so (20, 30) **is** in the feasible set.

The vertices of the feasible set are (40, 0), (100/3, 100/6), (20, 30), and (0, 40). Evaluate the objective function at each vertex:

(40, 0): 20(40) + 26(0) = 800

(100/3, 100/6): 20(100/3) + 26(100/6) = 1100

(20, 30): 20(20) + 26(30) = 1180

(0, 40): 20(0) + 26(40) = 1040

The maximum profit is $1180, when $x_1 = 20$ widgets and $x_2 = 30$ whammies.

**18.** Take any $\mathbf{p}$, $\mathbf{q}$ in $\mathscr{F}$. Then $A\mathbf{p} \le \mathbf{b}$, $A\mathbf{q} \le \mathbf{b}$, $\mathbf{p} \ge \mathbf{0}$, and $\mathbf{q} \ge \mathbf{0}$. Take any scalar $t$ such that $0 \le t \le 1$, and let $\mathbf{x} = (1-t)\mathbf{p} + t\mathbf{q}$. Then

$$A\mathbf{x} = A[(1-t)\mathbf{p} + t\mathbf{q}\,] = (1-t)A\mathbf{p} + tA\mathbf{q} \qquad (*)$$

by the linearity of matrix multiplication. Since $t$ and $1-t$ are both nonnegative, $(1-t)A\mathbf{p} \le (1-t)\mathbf{b}$ and $tA\mathbf{p} \le t\mathbf{b}$. Thus, the right side of (*) is less than or equal to $\mathbf{b}$. Also, $\mathbf{x} \ge 0$ because $\mathbf{p}$ and $\mathbf{q}$ have this property and the constants $(1-t)$ and $t$ are nonnegative. Thus, $\mathbf{x}$ is in $\mathscr{F}$. So the line segment between $\mathbf{p}$ and $\mathbf{q}$ is in $\mathscr{F}$. This proves that $\mathscr{F}$ is convex.

**19.** Take any $\mathbf{p}$ and $\mathbf{q}$ in $S$, with $\mathbf{p} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{q} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. Then $\mathbf{v}^T\mathbf{p} \le c$ and $\mathbf{v}^T\mathbf{q} \le c$. Take any scalar $t$ such that $0 \le t \le 1$. Then, by the linearity of matrix multiplication (or the dot product if $\mathbf{v}^T\mathbf{p}$ is written as $\mathbf{v} \cdot \mathbf{p}$, and so on),

$\mathbf{v}^T[(1-t)\mathbf{p} + t\mathbf{q}] = (1-t)\mathbf{v}^T\mathbf{p} + t\mathbf{v}^T\mathbf{q} \le (1-t)c + tc = c$

because $(1 - t)$ and $t$ are both positive and $\mathbf{p}$ and $\mathbf{q}$ are in $S$. So the line segment between $\mathbf{p}$ and $\mathbf{q}$ is in $S$. Since $\mathbf{p}$ and $\mathbf{q}$ were any points in $S$, the set $S$ is convex.

**20.** Let $S$ be the intersection of $S_1, \ldots, S_5$, and take $\mathbf{x}$ and $\mathbf{y}$ in $S$. Then $\mathbf{x}$ and $\mathbf{y}$ are in $S_i$ for $i = 1, \ldots, 5$. For any $t$, with $0 \le t \le 1$, and any $i$, with $1 \le i \le 5$, $(1-t)\mathbf{x} + t\mathbf{y}$ is in $S_i$ because $S_i$ is convex. Then $(1-t)\mathbf{x} + t\mathbf{y}$ is in $S$, by definition of the intersection. This proves that $S$ is a convex set.

**21.** Let $S = \{\mathbf{x} : f(\mathbf{x}) = d\}$, and take $\mathbf{p}$ and $\mathbf{q}$ in $S$. Also, take $t$ with $0 \le t \le 1$, and let $\mathbf{x} = (1-t)\mathbf{p} + t\mathbf{q}$. Then

$f(\mathbf{x}) = \mathbf{c}^T\mathbf{x} = \mathbf{c}^T[(1-t)\mathbf{p} + t\mathbf{q}] = (1-t)\mathbf{c}^T\mathbf{p} + \mathrm{t}\mathbf{c}^T\mathbf{q} = (1-t)d + td = d$

Thus, $\mathbf{x}$ is in $S$. This shows that $S$ is convex.

# 9.3 - Linear Programming – Simplex Method

**1.**
$$\begin{array}{c} \begin{array}{cccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & M & \end{array} \\ \left[\begin{array}{cccccc|c} 2 & 7 & 10 & 1 & 0 & 0 & 20 \\ 3 & 4 & 18 & 0 & 1 & 0 & 25 \\ \hline -21 & -25 & -15 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

**2.**
$$\begin{array}{c} \begin{array}{cccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & M & \end{array} \\ \left[\begin{array}{cccccc|c} 3 & 5 & 1 & 0 & 0 & 0 & 30 \\ 2 & 7 & 0 & 1 & 0 & 0 & 24 \\ 6 & 1 & 0 & 0 & 1 & 0 & 42 \\ \hline -22 & -14 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

**3. a.** $x_2$ **b.**
$$\begin{array}{c} \begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & M & \end{array} \\ \left[\begin{array}{ccccc|c} \frac{7}{2} & 0 & 1 & -\frac{1}{2} & 0 & 5 \\ \frac{3}{2} & 1 & 0 & \frac{1}{2} & 0 & 15 \\ \hline 11 & 0 & 0 & 5 & 1 & 150 \end{array}\right] \end{array}$$
**c.** $x_1 = 0,\ x_2 = 15,\ x_3 = 5,\ x_4 = 0,\ M = 150$ **d.** optimal

**4. a.** $x_1$ **b.**
$$\begin{array}{c} \begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & M & \end{array} \\ \left[\begin{array}{ccccc|c} 0 & 1 & 7 & 1 & 0 & 10 \\ 1 & 0 & 5 & 1 & 0 & 6 \\ \hline 0 & 0 & 28 & 5 & 1 & 47 \end{array}\right] \end{array}$$
**c.** $x_1 = 6,\ x_2 = 10,\ x_3 = 0,\ x_4 = 0,\ M = 47$ **d.** optimal

**5. a.** $x_1$ **b.**
$$\begin{array}{c} \begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & M & \end{array} \\ \left[\begin{array}{ccccc|c} 0 & 2 & 1 & -1 & 0 & 4 \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 8 \\ \hline 0 & -2 & 0 & 3 & 1 & 48 \end{array}\right] \end{array}$$
**c.** $x_1 = 8,\ x_2 = 0,\ x_3 = 4,\ x_4 = 0,\ M = 48$

**d.** not optimal

**6. a.** $x_2$ **b.**
$$\begin{array}{c} \begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & M & \end{array} \\ \left[\begin{array}{ccccc|c} -11 & 0 & 1 & -\frac{4}{3} & 0 & 40 \\ 2 & 1 & 0 & \frac{1}{6} & 0 & 5 \\ \hline 8 & 0 & 0 & \frac{1}{2} & 1 & 15 \end{array}\right] \end{array}$$
**c.** $x_1 = 0,\ x_2 = 5,\ x_3 = 40,\ x_4 = 0,\ M = 15$ **d.** optimal

**7.** False. A slack variable is used to change an inequality into an equality.

**8.** True. See the definition of a slack variable.

**9.** True. See the definition of a basic solution.

**10.** True. See the comment before Example 3.

**11.** False. The initial basic solution will be infeasible, but there may still be a basic feasible solution.

**12.** False. The bottom entry in the right column gives the current value of the objective function. It will be the maximum value only if the current solution is optimal.

**13.** First, bring $x_2$ into the solution; pivot with row 1. Then, bring $x_1$ into the solution; pivot with row 2. The maximum is 150, when $x_1 = 3$ and $x_2 = 10$.

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & M \end{array} \qquad \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & M \end{array} \qquad \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & M \end{array}$$

$$\left[\begin{array}{rrrrr|r} 2 & 3 & 1 & 0 & 0 & 36 \\ 5 & 4 & 0 & 1 & 0 & 55 \\ \hline -10 & -12 & 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{rrrrr|r} \frac{2}{3} & 1 & \frac{1}{3} & 0 & 0 & 12 \\ \frac{7}{3} & 0 & -\frac{4}{3} & 1 & 0 & 7 \\ \hline -2 & 0 & 4 & 0 & 1 & 144 \end{array}\right] \sim \left[\begin{array}{rrrrr|r} 0 & 1 & \frac{5}{7} & -\frac{2}{7} & 0 & 10 \\ 1 & 0 & -\frac{4}{7} & \frac{3}{7} & 0 & 3 \\ \hline 0 & 0 & \frac{20}{7} & \frac{6}{7} & 1 & 150 \end{array}\right]$$

**14.** First, bring $x_1$ into the solution; pivot with row 2. Next, scale row 1 to simplify the arithmetic. Finally, bring $x_2$ into the solution; pivot with row 1. The maximum is 98, when $x_1 = 10$ and $x_2 = 12$.

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & M \end{array} \qquad \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & M \end{array}$$

$$\left[\begin{array}{rrrrr|r} 1 & 5 & 1 & 0 & 0 & 70 \\ 3 & 2 & 0 & 1 & 0 & 54 \\ \hline -5 & -4 & 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{rrrrr|r} 0 & \frac{13}{3} & 1 & -\frac{1}{3} & 0 & 52 \\ 1 & \frac{2}{3} & 0 & \frac{1}{3} & 0 & 18 \\ \hline 0 & -\frac{2}{3} & 0 & \frac{5}{3} & 1 & 90 \end{array}\right]$$

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & M \end{array} \qquad \begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & M \end{array}$$

$$\sim \left[\begin{array}{rrrrr|r} 0 & 1 & \frac{3}{13} & -\frac{1}{13} & 0 & 12 \\ 1 & \frac{2}{3} & 0 & \frac{1}{3} & 0 & 18 \\ \hline 0 & -\frac{2}{3} & 0 & \frac{5}{3} & 1 & 90 \end{array}\right] \sim \left[\begin{array}{rrrrr|r} 0 & 1 & \frac{3}{13} & -\frac{1}{13} & 0 & 12 \\ 1 & 0 & -\frac{2}{13} & \frac{5}{13} & 0 & 10 \\ \hline 0 & 0 & \frac{2}{13} & \frac{21}{13} & 1 & 98 \end{array}\right]$$

**15.** First, bring $x_2$ into the solution; pivot with row 2. Then bring $x_1$ into the solution; pivot with row 3. The maximum is 56, when $x_1 = 9$ and $x_2 = 4$.

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & M \end{array} \qquad \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & M \end{array} \qquad \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & M \end{array}$$

$$\left[\begin{array}{rrrrrr|r} 1 & 2 & 1 & 0 & 0 & 0 & 26 \\ 2 & 3 & 0 & 1 & 0 & 0 & 30 \\ 1 & 1 & 0 & 0 & 1 & 0 & 13 \\ \hline -4 & -5 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{rrrrrr|r} -\frac{1}{3} & 0 & 1 & -\frac{2}{3} & 0 & 0 & 6 \\ \frac{2}{3} & 1 & 0 & \frac{1}{3} & 0 & 0 & 10 \\ \frac{1}{3} & 0 & 0 & -\frac{1}{3} & 1 & 0 & 3 \\ \hline -\frac{2}{3} & 0 & 0 & \frac{5}{3} & 0 & 1 & 50 \end{array}\right] \sim \left[\begin{array}{rrrrrr|r} 0 & 0 & 1 & -1 & 0 & 0 & 9 \\ 0 & 1 & 0 & 1 & -2 & 0 & 4 \\ 1 & 0 & 0 & -1 & 3 & 0 & 9 \\ \hline 0 & 0 & 0 & 1 & 2 & 1 & 56 \end{array}\right]$$

**16.** First, bring $x_2$ into the solution; pivot with row 3. Next, bring $x_1$ into the solution; pivot with row 1. Finally, bring $x_3$ into the solution; pivot with row 2. The maximum is 70, when $x_1 = 6$, $x_2 = 11$, and $x_3 = 1$.

$$\begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M & \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & 2 & 0 & 1 & 0 & 0 & 0 & 28 \\ 2 & 0 & 4 & 0 & 1 & 0 & 0 & 16 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 12 \\ \hline -2 & -5 & -3 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \sim \begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M & \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & 0 & -2 & 1 & 0 & -2 & 0 & 4 \\ 2 & 0 & 4 & 0 & 1 & 0 & 0 & 16 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 12 \\ \hline -2 & 0 & 2 & 0 & 0 & 5 & 1 & 60 \end{array}\right] \end{array}$$

$$\sim \begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M & \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & 0 & -2 & 1 & 0 & -2 & 0 & 4 \\ 0 & 0 & 8 & -2 & 1 & 4 & 0 & 8 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 12 \\ \hline 0 & 0 & -2 & 2 & 0 & 1 & 1 & 68 \end{array}\right] \end{array} \sim \begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M & \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{4} & -1 & 0 & 6 \\ 0 & 0 & 1 & -\frac{1}{4} & \frac{1}{8} & \frac{1}{2} & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{8} & \frac{1}{2} & 0 & 11 \\ \hline 0 & 0 & 0 & \frac{3}{2} & \frac{1}{4} & 2 & 1 & 70 \end{array}\right] \end{array}$$

**17.** Convert this to a maximization problem for $-12x_1 - 5x_2$, and reverse the first constraint inequality. Beginning with the first tableau below, bring $x_1$ into the solution, using row 1 as the pivot row. Then bring $x_2$ into the solution; pivot with row 2. The maximum value of $-12x_1 - 5x_2$ is $-180$, so the minimum of the original objective function $12x_1 + 5x_2$ is 180, when $x_1$ is 10 and $x_2$ is 12.

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & M & \end{array} \\ \left[\begin{array}{ccccc|c} -2 & -1 & 1 & 0 & 0 & -32 \\ -3 & 5 & 0 & 1 & 0 & 30 \\ \hline 12 & 5 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \sim \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & M & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 16 \\ 0 & \frac{13}{2} & -\frac{3}{2} & 1 & 0 & 78 \\ \hline 0 & -1 & 6 & 0 & 1 & -192 \end{array}\right] \end{array} \sim \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & M & \end{array} \\ \left[\begin{array}{ccccc|c} 1 & 0 & -\frac{5}{13} & -\frac{1}{13} & 0 & 10 \\ 0 & 1 & -\frac{3}{13} & \frac{2}{13} & 0 & 12 \\ \hline 0 & 0 & \frac{75}{13} & \frac{2}{13} & 1 & -180 \end{array}\right] \end{array}$$

**18.** Convert this to a maximization problem for $-2x_1 - 3x_2 - 3x_3$, and reverse the first two constraint inequalities. Beginning with the first tableau below, bring $x_3$ into the solution, with row 2 as the pivot. Then bring $x_2$ into the solution; pivot with row 1. The maximum is $-33$, so the minimum of $2x_1 + 3x_2 + 3x_3$ is 33, when $x_1 = 0$, $x_2 = 4$, and $x_3 = 7$.

$$\begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M & \end{array} \\ \left[\begin{array}{ccccccc|c} -1 & 2 & 0 & 1 & 0 & 0 & 0 & 8 \\ 0 & -2 & -1 & 0 & 1 & 0 & 0 & -15 \\ 2 & -1 & 1 & 0 & 0 & 1 & 0 & 25 \\ \hline 2 & 3 & 3 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \sim \begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M & \end{array} \\ \left[\begin{array}{ccccccc|c} -1 & 2 & 0 & 1 & 0 & 0 & 0 & 8 \\ 0 & 2 & 1 & 0 & -1 & 0 & 0 & 15 \\ 2 & -3 & 0 & 0 & 1 & 1 & 0 & 10 \\ \hline 2 & -3 & 0 & 0 & 3 & 0 & 1 & -45 \end{array}\right] \end{array}$$

$$\sim \begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M & \end{array} \\ \left[\begin{array}{ccccccc|c} -\frac{1}{2} & 1 & 0 & \frac{1}{2} & 0 & 0 & 0 & 4 \\ 1 & 0 & 1 & -1 & -1 & 0 & 0 & 7 \\ \frac{1}{2} & 0 & 0 & \frac{3}{2} & 1 & 1 & 0 & 22 \\ \hline \frac{1}{2} & 0 & 0 & \frac{3}{2} & 3 & 0 & 1 & -33 \end{array}\right] \end{array}$$

**19.** Begin with the same initial simplex tableau, bringing $x_1$ into the solution, with row 1 as the pivot row. This makes $b_2$ equal to – 12, so the basic solution is still not feasible. To correct this, pivot on the negative entry –2 in the second column. This brings $x_2$ into the solution. The tableau now shows the optimal solution. The maximum of $-x_1 - 2x_2$ is $-20$, so the minimum of $x_1 + 2x_2$ is 20, when $x_1 = 8$ and $x_2 = 6$.

$$\begin{array}{c}\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & M & \end{array}\\ \left[\begin{array}{ccccc|c} -1 & -1 & 1 & 0 & 0 & -14 \\ 1 & -1 & 0 & 1 & 0 & 2 \\ \hline 1 & 2 & 0 & 0 & 1 & 0 \end{array}\right]\end{array} \sim \begin{array}{c}\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & M & \end{array}\\ \left[\begin{array}{ccccc|c} 1 & 1 & -1 & 0 & 0 & 14 \\ 0 & -2 & 1 & 1 & 0 & -12 \\ \hline 0 & 1 & 1 & 0 & 1 & -14 \end{array}\right]\end{array} \sim \begin{array}{c}\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & M & \end{array}\\ \left[\begin{array}{ccccc|c} 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & 0 & 6 \\ 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 8 \\ \hline 0 & 0 & \frac{3}{2} & \frac{1}{2} & 1 & -20 \end{array}\right]\end{array}$$

**20.** From the bottom row of the tableau, $x_1$ must be brought into the solution first. The ratios to consider are 12/1 in row 1 and 0/1 in row 2. So pivot with row 2. Next, bring $x_2$ into the solution; pivot with row 3 (because the ratio 0/1 is less than the ratio 12/2). Finally, bring $x_3$ into the solution; pivot with row 1. The maximum annual income of \$1,100 is provided by \$6,000 in mutual funds, \$4,000 in CDs, and \$2,000 in savings.

$$\begin{array}{c}\begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M & \end{array}\\ \left[\begin{array}{ccccccc|c} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 12{,}000 \\ 1 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 & 0 & 0 \\ \hline -.11 & -.08 & -.06 & 0 & 0 & 0 & 1 & 0 \end{array}\right]\end{array} \sim \begin{array}{c}\begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M & \end{array}\\ \left[\begin{array}{ccccccc|c} 0 & 2 & 2 & 1 & -1 & 0 & 0 & 12{,}000 \\ 1 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & -.19 & -.17 & 0 & .11 & 0 & 1 & 0 \end{array}\right]\end{array}$$

$$\sim \begin{array}{c}\begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M & \end{array}\\ \left[\begin{array}{ccccccc|c} 0 & 0 & 6 & 1 & -1 & -2 & 0 & 12{,}000 \\ 1 & 0 & -3 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & -.55 & 0 & .11 & .19 & 1 & 0 \end{array}\right]\end{array} \sim \begin{array}{c}\begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M & \end{array}\\ \left[\begin{array}{ccccccc|c} 0 & 0 & 1 & \frac{1}{6} & -\frac{1}{6} & -\frac{1}{3} & 0 & 2{,}000 \\ 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 6{,}000 \\ 0 & 1 & 0 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 0 & 4{,}000 \\ \hline 0 & 0 & 0 & \frac{11}{120} & \frac{11}{600} & \frac{1}{150} & 1 & 1{,}100 \end{array}\right]\end{array}$$

**21.** The simplex tableau below is based on the problem of the Benri Company (Exercise 17 in Section 9.2) to maximize the profit function $20x_1 + 26x_2$ subject to various amounts of labor available for the three-step production process. To begin the simplex method, bring $x_2$ into the solution; pivot with row 2. Then, bring $x_1$ into the solution; pivot with row 3. The profit is maximized at \$1180, by making 20 widgets and 30 whammies each day.

$$\begin{array}{c}\begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & M & \end{array}\\ \left[\begin{array}{cccccc|c} 5 & 2 & 1 & 0 & 0 & 0 & 200 \\ \frac{1}{5} & \frac{2}{5} & 0 & 1 & 0 & 0 & 16 \\ \frac{1}{5} & \frac{1}{5} & 0 & 0 & 1 & 0 & 10 \\ \hline -20 & -26 & 0 & 0 & 0 & 1 & 0 \end{array}\right]\end{array} \sim \begin{array}{c}\begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & M & \end{array}\\ \left[\begin{array}{cccccc|c} 4 & 0 & 1 & -5 & 0 & 0 & 120 \\ \frac{1}{2} & 1 & 0 & \frac{5}{2} & 0 & 0 & 40 \\ \frac{1}{10} & 0 & 0 & -\frac{1}{2} & 1 & 0 & 2 \\ \hline -7 & 0 & 0 & 65 & 0 & 1 & 1040 \end{array}\right]\end{array}$$

$$\sim \begin{array}{c}\begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & M & \end{array}\\ \left[\begin{array}{cccccc|c} 0 & 0 & 1 & 15 & -40 & 0 & 40 \\ 0 & 1 & 0 & 5 & -5 & 0 & 30 \\ 1 & 0 & 0 & -5 & 10 & 0 & 20 \\ \hline 0 & 0 & 0 & 30 & 0 & 1 & 1180 \end{array}\right]\end{array}$$

**22.** The simplex tableau below is based on the summary at the end of Example 1 in Section 9.2. To begin the simplex method, bring $x_2$ into the solution; pivot with row 2. Then bring $x_1$ into the solution; pivot with row 3. The \$1250 maximum is achieved when $x_1 = 100$ (bags of EverGreen) and $x_2 = 350$ (bags of QuickGreen).

$$\sim \begin{array}{c} \begin{array}{cccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & M & \end{array} \\ \left[\begin{array}{cccccc|c} 0 & 0 & 1 & 1 & -4 & 0 & 200 \\ 0 & 1 & 0 & 1 & -1 & 0 & 350 \\ 1 & 0 & 0 & -1 & 2 & 0 & 100 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 & 1250 \end{array}\right] \end{array}$$

# 9.4 - Duality

**1.** Minimize $36y_1 + 55y_2$
subject to
$2y_1 + 5y_2 \geq 10$
$3y_1 + 4y_2 \geq 12$
and $y_1 \geq 0, y_2 \geq 0$.

**2.** Minimize $70y_1 + 54y_2$
subject to
$y_1 + 3y_2 \geq 5$
$5y_1 + 2y_2 \geq 4$
and $y_1 \geq 0, y_2 \geq 0$.

**3.** Minimize $26y_1 + 30y_2 + 13y_3$
subject to
$y_1 + 2y_2 + y_3 \geq 4$
$2y_1 + 3y_2 + y_3 \geq 5$
and $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$.

**4.** Minimize $28y_1 + 16y_2 + 12y_3$
subject to
$y_1 + 2y_2 \geq 2$
$2y_1 + y_3 \geq 5$
$4y_2 + y_3 \geq 3$
and $y_1 \geq 0, y_2 \geq 0, y_3 \geq 0$.

**5.** The final tableau from Exercise 9 in Section 9.3 is

$$\begin{array}{c} \begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & M & \end{array} \\ \left[\begin{array}{ccccc|c} 0 & 1 & \frac{5}{7} & -\frac{2}{7} & 0 & 10 \\ 1 & 0 & -\frac{4}{7} & \frac{3}{7} & 0 & 3 \\ \hline 0 & 0 & \frac{20}{7} & \frac{6}{7} & 1 & 150 \end{array}\right] \end{array}$$

The solution of the dual problem is displayed by the entries in row 3 of columns 3, 4, and 6. The minimum is $M = 150$, attained when $y_1 = \frac{20}{7}$ and $y_2 = \frac{6}{7}$.

**6.** The final tableau from Exercise 10 in Section 9.3 is

$$\begin{array}{c} \begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & M & \end{array} \\ \left[\begin{array}{ccccc|c} 0 & 1 & \frac{3}{13} & -\frac{1}{13} & 0 & 12 \\ 1 & 0 & -\frac{2}{13} & \frac{5}{13} & 0 & 10 \\ \hline 0 & 0 & \frac{2}{13} & \frac{21}{13} & 1 & 98 \end{array}\right] \end{array}$$

The solution of the dual problem is displayed by the entries in row 3 of columns 3, 4, and 6. The minimum is $M = 98$, attained when $y_1 = \frac{2}{13}$ and $y_2 = \frac{21}{13}$.

**7.** The final tableau from Exercise 11 in Section 9.3 is

$$\begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & M \end{array} \\ \left[\begin{array}{cccccc|c} 0 & 0 & 1 & -1 & 0 & 0 & 9 \\ 0 & 1 & 0 & 1 & -2 & 0 & 4 \\ 1 & 0 & 0 & -1 & 3 & 0 & 9 \\ \hline 0 & 0 & 0 & 1 & 2 & 1 & 56 \end{array}\right] \end{array}$$

The solution of the dual problem is displayed by the entries in row 4 of columns 3, 4, 5, and 7. The minimum is $M = 56$, attained when $y_1 = 0$, $y_2 = 1$, and $y_3 = 2$.

**8.** The final tableau from Exercise 12 in Section 9.3 is

$$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{4} & -1 & 0 & 6 \\ 0 & 0 & 1 & -\frac{1}{4} & \frac{1}{8} & \frac{1}{2} & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{8} & \frac{1}{2} & 0 & 11 \\ \hline 0 & 0 & 0 & \frac{3}{2} & \frac{1}{4} & 2 & 1 & 70 \end{array}\right] \end{array}$$

The solution of the dual problem is displayed by the entries in row 4 of columns 4, 5, 6, and 8. The minimum is $M = 70$, attained when $y_1 = \frac{3}{2}$, $y_2 = \frac{1}{4}$, and $y_3 = 2$.

**9.** False. It should be $A^T\mathbf{y} \geq \mathbf{c}$.

**10.** True. See the comment before Theorem 7.

**11.** True. Theorem 7.

**12.** True. Theorem 7.

**13.** True. Theorem 7.

**14.** True. Theorem 7.

**15** False. The marginal value is zero if it is in the optimal solution. See Example 4.

**16.** False. The coordinates of **u** and **v** are equal to one. The vectors do not have length one.

**17.** The dual problem is to maximize $4y_1 + 5y_2$ subject to $\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \leq \begin{bmatrix} 16 \\ 10 \\ 20 \end{bmatrix}$ and $\mathbf{y} \geq \mathbf{0}$.

Solve the dual problem with the simplex method:

$$\begin{array}{c} \begin{array}{cccccc} y_1 & y_2 & y_3 & y_4 & y_5 & M \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 2 & 1 & 0 & 0 & 0 & 16 \\ 1 & 1 & 0 & 1 & 0 & 0 & 10 \\ 3 & 2 & 0 & 0 & 1 & 0 & 20 \\ \hline -4 & -5 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \sim \begin{array}{c} \begin{array}{cccccc} y_1 & y_2 & y_3 & y_4 & y_5 & M \end{array} \\ \left[\begin{array}{cccccc|c} \frac{1}{2} & 1 & \frac{1}{2} & 0 & 0 & 0 & 8 \\ \frac{1}{2} & 0 & -\frac{1}{2} & 1 & 0 & 0 & 2 \\ 2 & 0 & -1 & 0 & 1 & 0 & 4 \\ \hline -\frac{3}{2} & 0 & \frac{5}{2} & 0 & 0 & 1 & 40 \end{array}\right] \end{array}$$

$$\sim \begin{array}{c} \begin{array}{cccccc|c} y_1 & y_2 & y_3 & y_4 & y_5 & M & \end{array} \\ \left[\begin{array}{cccccc|c} 0 & 1 & \frac{3}{4} & 0 & -\frac{1}{4} & 0 & 7 \\ 0 & 0 & -\frac{1}{4} & 1 & -\frac{1}{4} & 0 & 1 \\ 1 & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & 2 \\ \hline 0 & 0 & \frac{7}{4} & 0 & \frac{3}{4} & 1 & 43 \end{array}\right] \end{array}$$

The solution of the dual of the dual (the primal) is $x_1 = \frac{7}{4}$, $x_2 = 0$, $x_3 = \frac{3}{4}$, with $M = 43$.

**18.** The dual problem is to maximize $3y_1 + 4y_2 + 2y_3$ subject to

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \le \begin{bmatrix} 10 \\ 14 \end{bmatrix}$$ and $\mathbf{y} \ge \mathbf{0}$. Use the simplex tableau for the dual problem:

$$\begin{array}{c} \begin{array}{cccccc|c} y_1 & y_2 & y_3 & y_4 & y_5 & M & \end{array} \\ \left[\begin{array}{cccccc|c} 1 & 2 & 3 & 1 & 0 & 0 & 10 \\ 2 & 1 & 1 & 0 & 1 & 0 & 14 \\ \hline -3 & -4 & -2 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \sim \begin{array}{c} \begin{array}{cccccc|c} y_1 & y_2 & y_3 & y_4 & y_5 & M & \end{array} \\ \left[\begin{array}{cccccc|c} \frac{1}{2} & 1 & \frac{3}{2} & \frac{1}{2} & 0 & 0 & 5 \\ \frac{3}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} & 1 & 0 & 9 \\ \hline -1 & 0 & 4 & 2 & 0 & 1 & 20 \end{array}\right] \end{array}$$

$$\sim \begin{array}{c} \begin{array}{cccccc|c} y_1 & y_2 & y_3 & y_4 & y_5 & M & \end{array} \\ \left[\begin{array}{cccccc|c} \frac{1}{2} & 1 & \frac{3}{2} & \frac{1}{2} & 0 & 0 & 5 \\ 1 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 6 \\ \hline -1 & 0 & 4 & 2 & 0 & 1 & 20 \end{array}\right] \end{array} \sim \begin{array}{c} \begin{array}{cccccc|c} y_1 & y_2 & y_3 & y_4 & y_5 & M & \end{array} \\ \left[\begin{array}{cccccc|c} 0 & 1 & \frac{5}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 2 \\ 1 & 0 & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 6 \\ \hline 0 & 0 & \frac{11}{3} & \frac{5}{3} & \frac{2}{3} & 1 & 26 \end{array}\right] \end{array}$$

The solution of the dual of the dual (the primal) is $x_1 = \frac{5}{3}$, $x_2 = \frac{2}{3}$, with the minimum $M = 26$.

**19.** The problem in Exercise 2 of Section 9.2 is to minimize $\mathbf{c}^T\mathbf{x}$ subject to $A\mathbf{x} \ge \mathbf{b}$ and $\mathbf{x} \ge \mathbf{0}$, where $\mathbf{x}$ lists the number of bags of Pixie Power and Misty Might, and

$$\mathbf{c} = \begin{bmatrix} 50 \\ 40 \end{bmatrix}, \ A = \begin{bmatrix} 3 & 2 \\ 2 & 4 \\ 1 & 3 \\ 2 & 1 \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 28 \\ 30 \\ 20 \\ 25 \end{bmatrix}, \text{ and } \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

The dual of a minimization problem involving a matrix is a maximization problem involving the transpose of the matrix, with the vector data for the objective function and the constraint equation interchanged. Since the notation was established in Exercise 2 for a minimization problem, the notation here is "reversed" from the usual notation for a primal problem. Thus, the dual of the primal problem stated above is to maximize $\mathbf{b}^T\mathbf{y}$ subject to $A^T\mathbf{y} \le \mathbf{c}$ and $\mathbf{y} \ge \mathbf{0}$. That is,

maximize $28y_1 + 30y_2 + 20y_3 + 25y_4$ subject to $\begin{bmatrix} 3 & 2 & 1 & 2 \\ 2 & 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \le \begin{bmatrix} 50 \\ 40 \end{bmatrix}$

Here are the simplex calculations for this dual problem:

$$\begin{array}{c}\begin{array}{cccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & M & \\ \end{array}\\ \left[\begin{array}{ccccccc|c} 3 & 2 & 1 & 2 & 1 & 0 & 0 & 50 \\ 2 & 4 & 3 & 1 & 0 & 1 & 0 & 40 \\ \hline -28 & -30 & -20 & -25 & 0 & 0 & 1 & 0 \end{array}\right]\end{array} \sim \begin{array}{c}\begin{array}{cccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & M & \\ \end{array}\\ \left[\begin{array}{ccccccc|c} 2 & 0 & -\frac{1}{2} & \frac{3}{2} & 1 & -\frac{1}{2} & 0 & 30 \\ \frac{1}{2} & 1 & \frac{3}{4} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 10 \\ \hline -13 & 0 & \frac{5}{2} & -\frac{35}{2} & 0 & \frac{15}{2} & 1 & 300 \end{array}\right]\end{array}$$

$$\sim \begin{array}{c}\begin{array}{cccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & M & \\ \end{array}\\ \left[\begin{array}{ccccccc|c} \frac{4}{3} & 0 & -\frac{1}{3} & 1 & \frac{2}{3} & -\frac{1}{3} & 0 & 20 \\ \frac{1}{6} & 1 & \frac{5}{6} & 0 & -\frac{1}{6} & \frac{1}{3} & 0 & 5 \\ \hline \frac{31}{3} & 0 & -\frac{10}{3} & 0 & \frac{35}{3} & \frac{5}{3} & 1 & 650 \end{array}\right]\end{array} \sim \begin{array}{c}\begin{array}{cccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & M & \\ \end{array}\\ \left[\begin{array}{ccccccc|c} \frac{7}{5} & \frac{2}{5} & 0 & 1 & \frac{3}{5} & -\frac{1}{5} & 0 & 22 \\ \frac{1}{5} & \frac{6}{5} & 1 & 0 & -\frac{1}{5} & \frac{2}{5} & 0 & 6 \\ \hline 11 & 4 & 0 & 0 & 11 & 3 & 1 & 670 \end{array}\right]\end{array}$$

Since the solution of the original problem is the dual of the problem solved by the simplex method, the solution is given by the slack variables $y_5 = 11$ and $y_6 = 3$. The value of the objective is the same for the primal and dual problems, so the minimum cost is \$670. This is achieved by blending 11 bags of PixiePower and 3 bags of MistyMight.

**20.** Express costs in thousands of dollars, let $x_1$ be the number of days refinery A operates, and let $x_2$ be the number of days refinery B operates. Then the problem in Example 2 of Section 9.2 is to minimize $3.5x_1 + 3x_2$ subject to $\begin{bmatrix} 12 & 4 \\ 4 & 4 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} 48 \\ 32 \\ 20 \end{bmatrix}$. The dual problem is to maximize $48y_1 + 32y_2 + 20y_3$ subject to $\begin{bmatrix} 12 & 4 & 1 \\ 4 & 4 & 5 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \leq \begin{bmatrix} 3.5 \\ 3 \end{bmatrix}$.

Use the simplex tableau for this dual problem. The first pivot is on $y_1$, because the entry $-48$ is the most negative entry in the bottom row. The first row is chosen because the ratio $b_1/a_{11}$ is smaller than $b_2/a_{21}$.

$$\begin{array}{c}\begin{array}{ccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & M & \\ \end{array}\\ \left[\begin{array}{cccccc|c} 12 & 4 & 1 & 1 & 0 & 0 & 3.5 \\ 4 & 4 & 5 & 0 & 1 & 0 & 3 \\ \hline -48 & -32 & -20 & 0 & 0 & 1 & 0 \end{array}\right]\end{array} \sim \begin{array}{c}\begin{array}{ccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & M & \\ \end{array}\\ \left[\begin{array}{cccccc|c} 1 & \frac{1}{3} & \frac{1}{12} & \frac{1}{12} & 0 & 0 & \frac{7}{24} \\ 0 & \frac{8}{3} & \frac{14}{3} & -\frac{1}{3} & 1 & 0 & \frac{11}{6} \\ \hline 0 & -16 & -16 & 4 & 0 & 1 & 14 \end{array}\right]\end{array}$$

Now, two negative entries in the bottom row happen to be equal, so either $y_2$ or $y_3$ can be the next pivot. When $y_2$ is used, the result is

$$\sim \begin{array}{c}\begin{array}{ccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & M & \\ \end{array}\\ \left[\begin{array}{cccccc|c} 1 & 0 & -\frac{1}{2} & \frac{1}{8} & -\frac{1}{8} & 0 & \frac{1}{16} \\ 0 & 1 & \frac{7}{4} & -\frac{1}{8} & \frac{3}{8} & 0 & \frac{11}{16} \\ \hline 0 & 0 & 12 & 2 & 6 & 1 & 25 \end{array}\right]\end{array}$$

When $y_3$ is used as a pivot in the second tableau above, more work is required:

$$\begin{array}{c} \begin{array}{ccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & M & \end{array} \\ \sim\left[\begin{array}{cccccc|c} 1 & \frac{2}{7} & 0 & \frac{5}{56} & -\frac{1}{56} & 0 & \frac{29}{112} \\ 0 & \frac{4}{7} & 1 & -\frac{1}{14} & \frac{3}{14} & 0 & \frac{11}{28} \\ \hline 0 & -\frac{48}{7} & 0 & 4 & \frac{24}{7} & 1 & \frac{142}{7} \end{array}\right] \end{array} \begin{array}{c} \begin{array}{ccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & M & \end{array} \\ \sim\left[\begin{array}{cccccc|c} 1 & 0 & -\frac{1}{2} & \frac{1}{8} & -\frac{1}{8} & 0 & \frac{1}{16} \\ 0 & 1 & \frac{7}{4} & -\frac{1}{8} & \frac{3}{8} & 0 & \frac{11}{16} \\ \hline 0 & 0 & 12 & 2 & 6 & 1 & 25 \end{array}\right] \end{array}$$

An extra pivot operation is required because pivoting on $y_3$ increases $M$ by less than pivoting on $y_2$. This can be seen in advance, but the situation occurs so rarely, that a rule for deciding which pivot column to choose is hardly worth remembering. Notice that if $y_2$ is to be the pivot variable, then the row for this pivot is the one for which the ratio $b_i/a_{i2}$ is the smallest. (In this example, that ratio is $\frac{11}{6}\div\frac{8}{3}=\frac{11}{16}$.) If $y_3$ is the pivot variable, then the row for this pivot is the one for which the ratio $b_i/a_{i3}$ is the smallest. (In this example, that ratio is $\frac{11}{6}\div\frac{14}{3}=\frac{11}{28}$.) The rule is to choose the variable for which this "smallest" ratio is larger. In this case, since $\frac{11}{16}$ is larger than $\frac{11}{28}$, $y_2$ is the better choice for the pivot. Since so many ratios have to be computed, it seems easier just to pick either $y_2$ or $y_3$ and calculate the next tableau.

**21.** The marginal value is zero. This corresponds to labor in the fabricating department being under-utilized. That is, at the optimal production schedule with $x_1 = 20$ and $x_2 = 30$, only 160 of the 200 available hours in fabricating are needed. Thus, any extra labor there would be wasted: that is, i.e., it has value zero.

**22.** Allocate the additional hour of labor to the shipping department, thereby increasing the profit by \$70. The profit would increase by only \$30 if the hour of labor were added to packing, and not at all if the hour were added to fabricating.

**23.** $\hat{\mathbf{x}}=\begin{bmatrix}\frac{2}{3}\\0\\\frac{1}{3}\end{bmatrix},\ \hat{\mathbf{y}}=\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\end{bmatrix},\ v=1$

**24.** $\hat{\mathbf{x}}=\begin{bmatrix}\frac{1}{4}\\\frac{3}{4}\\0\end{bmatrix},\ \hat{\mathbf{y}}=\begin{bmatrix}\frac{3}{4}\\\frac{1}{4}\end{bmatrix},\ v=\frac{1}{4}$

**25.** The game is $\begin{bmatrix}1&2&-2\\0&1&4\\3&-1&1\end{bmatrix}$. Add 3 to shift the game: $\begin{bmatrix}4&5&1\\3&4&7\\6&2&4\end{bmatrix}$.

The linear programming tableau for this game is

$$\begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M & \end{array} \\ \left[\begin{array}{ccccccc|c} 4 & 5 & 1 & 1 & 0 & 0 & 0 & 1 \\ 3 & 4 & 7 & 0 & 1 & 0 & 0 & 1 \\ 6 & 2 & 4 & 0 & 0 & 1 & 0 & 1 \\ \hline -1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

Pivots:

$$\left[\begin{array}{ccccccc|c} 0 & \frac{11}{3} & -\frac{5}{3} & 1 & 0 & -\frac{2}{3} & 0 & \frac{1}{3} \\ 0 & 3 & 5 & 0 & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 1 & \frac{1}{3} & \frac{2}{3} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} \\ \hline 0 & -\frac{2}{3} & -\frac{1}{3} & 0 & 0 & \frac{1}{6} & 1 & \frac{1}{6} \end{array}\right] \sim \left[\begin{array}{ccccccc|c} 0 & 1 & -\frac{5}{11} & \frac{3}{11} & 0 & -\frac{2}{11} & 0 & \frac{1}{11} \\ 0 & 0 & \frac{70}{11} & -\frac{9}{11} & 1 & \frac{1}{22} & 0 & \frac{5}{22} \\ 1 & 0 & \frac{9}{11} & -\frac{1}{11} & 0 & \frac{5}{22} & 0 & \frac{3}{22} \\ \hline 0 & 0 & -\frac{7}{11} & \frac{2}{11} & 0 & \frac{1}{22} & 1 & \frac{5}{22} \end{array}\right]$$

$$\sim \left[\begin{array}{ccccccc|c} 0 & 1 & 0 & \frac{3}{14} & \frac{1}{14} & -\frac{5}{28} & 0 & \frac{3}{28} \\ 0 & 0 & 1 & -\frac{9}{70} & \frac{11}{70} & \frac{1}{140} & 0 & \frac{1}{28} \\ 1 & 0 & 0 & \frac{1}{70} & -\frac{9}{70} & \frac{31}{140} & 0 & \frac{3}{28} \\ \hline 0 & 0 & 0 & \frac{1}{10} & \frac{1}{10} & \frac{1}{20} & 1 & \frac{1}{4} \end{array}\right]$$

The optimal solution of the primal and dual problems, respectively, are

$\bar{y}_1 = \frac{3}{28},\ \bar{y}_2 = \frac{3}{28},\ \bar{y}_3 = \frac{1}{28}$, and $\bar{x}_1 = \frac{1}{10}, \bar{x}_2 = \frac{1}{10}, \bar{x}_3 = \frac{1}{20}$, with $\lambda = \frac{1}{4}$

The corresponding optimal mixed strategies for the column and row players, respectively, are:

$$\hat{\mathbf{y}} = \bar{\mathbf{y}}/\lambda = \bar{\mathbf{y}} \cdot 4 = \begin{bmatrix} \frac{3}{7} \\ \frac{3}{7} \\ \frac{1}{7} \end{bmatrix} \text{ and } \hat{\mathbf{x}} = \bar{\mathbf{x}}/\lambda = \bar{\mathbf{x}} \cdot 4 = \begin{bmatrix} \frac{2}{5} \\ \frac{2}{5} \\ \frac{1}{5} \end{bmatrix}$$

The value of the game with the shifted payoff matrix is 1/λ, which is 4, so the value of original game is 4 – 3 = 1.

**26.** The game is $\begin{bmatrix} 2 & 0 & 1 & -1 \\ -1 & 1 & -2 & 0 \\ 1 & -2 & 2 & 1 \end{bmatrix}$. Add 3 to shift the game: $\begin{bmatrix} 5 & 3 & 4 & 2 \\ 2 & 4 & 1 & 3 \\ 4 & 1 & 5 & 4 \end{bmatrix}$.

The linear programming tableau for this game is

$$\begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & M \end{array} \\ \left[\begin{array}{cccccccc|c} 5 & 3 & 4 & 2 & 1 & 0 & 0 & 0 & 1 \\ 2 & 4 & 1 & 3 & 0 & 1 & 0 & 0 & 1 \\ 4 & 1 & 5 & 4 & 0 & 0 & 1 & 0 & 1 \\ \hline -1 & -1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

The simplex method produces

$$\begin{array}{c} \begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & M \end{array} \\ \left[\begin{array}{cccccccc|c} \frac{49}{47} & 0 & 1 & 0 & \frac{13}{47} & -\frac{10}{47} & \frac{1}{47} & 0 & \frac{4}{47} \\ \frac{27}{47} & 1 & 0 & 0 & \frac{11}{47} & \frac{6}{47} & -\frac{10}{47} & 0 & \frac{7}{47} \\ -\frac{21}{47} & 0 & 0 & 1 & -\frac{19}{47} & \frac{11}{47} & \frac{13}{47} & 0 & \frac{5}{47} \\ \hline \frac{8}{47} & 0 & 0 & 0 & \frac{5}{47} & \frac{7}{47} & \frac{4}{47} & 1 & \frac{16}{47} \end{array}\right] \end{array}$$

The optimal solution of the primal and dual problems, respectively, are

$\bar{y}_1 = 0, \bar{y}_2 = \frac{7}{47}, \bar{y}_3 = \frac{4}{47}, \bar{y}_4 = \frac{5}{47}$, and $\bar{x}_1 = \frac{5}{47}, \bar{x}_2 = \frac{7}{47}, \bar{x}_3 = \frac{4}{47}$, with $\lambda = \frac{16}{47}$

The corresponding optimal mixed strategies for the column and row players, respectively, are

$$\hat{\mathbf{y}} = \overline{\mathbf{y}}/\lambda = \overline{\mathbf{y}} \cdot \tfrac{47}{16} = \begin{bmatrix} 0 \\ \frac{7}{16} \\ \frac{4}{16} \\ \frac{5}{16} \end{bmatrix} \quad \text{and} \quad \hat{\mathbf{x}} = \overline{\mathbf{x}}/\lambda = \overline{\mathbf{x}} \cdot \tfrac{47}{16} = \begin{bmatrix} \frac{5}{16} \\ \frac{7}{16} \\ \frac{4}{16} \end{bmatrix}$$

The value of the game with the shifted payoff matrix is $1/\lambda$, which is $\frac{47}{16}$, so the value of original game is $\frac{47}{16} - 3 = -\frac{1}{16}$.

**27.** The game is $\begin{bmatrix} 4 & 1 & -2 \\ 1 & 3 & 0 \\ -1 & 0 & 4 \end{bmatrix}$. Add 3 to shift the game: $\begin{bmatrix} 7 & 4 & 1 \\ 4 & 6 & 3 \\ 2 & 3 & 7 \end{bmatrix}$.

The linear programming problem is to maximize $y_1 + y_2 + y_3$ subject to

$$\begin{bmatrix} 7 & 4 & 1 \\ 4 & 6 & 3 \\ 2 & 3 & 7 \end{bmatrix} \le \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The tableau for this game is

$$\begin{array}{c} \begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M \end{array} \\ \left[\begin{array}{ccccccc|c} 7 & 4 & 1 & 1 & 0 & 0 & 0 & 1 \\ 4 & 6 & 3 & 0 & 1 & 0 & 0 & 1 \\ 2 & 3 & 7 & 0 & 0 & 1 & 0 & 1 \\ \hline -1 & -1 & -1 & 0 & 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

The simplex calculations are

$$\begin{array}{c} \begin{array}{ccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & M \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & \frac{4}{7} & \frac{1}{7} & \frac{1}{7} & 0 & 0 & 0 & \frac{1}{7} \\ 0 & \frac{26}{7} & \frac{14}{7} & -\frac{4}{7} & 1 & 0 & 0 & \frac{3}{7} \\ 0 & \frac{13}{7} & \frac{47}{7} & -\frac{2}{7} & 0 & 1 & 0 & \frac{5}{7} \\ \hline 0 & -\frac{3}{7} & -\frac{6}{7} & \frac{1}{7} & 0 & 0 & 1 & \frac{1}{7} \end{array}\right] \end{array} \sim \begin{array}{c} \begin{array}{ccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & M \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & \frac{20}{47} & 0 & \frac{7}{47} & 0 & -\frac{1}{47} & 0 & \frac{6}{47} \\ 0 & \frac{143}{47} & 0 & -\frac{22}{47} & 1 & -\frac{17}{47} & 0 & \frac{8}{47} \\ 0 & \frac{13}{47} & 1 & -\frac{2}{47} & 0 & \frac{7}{47} & 0 & \frac{5}{47} \\ \hline 0 & -\frac{4}{47} & 0 & \frac{5}{47} & 0 & \frac{6}{47} & 1 & \frac{11}{47} \end{array}\right] \end{array}$$

$$\sim \begin{array}{c} \begin{array}{ccccccc} y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & M \end{array} \\ \left[\begin{array}{ccccccc|c} 1 & 0 & 0 & \frac{3}{13} & -\frac{25}{143} & \frac{6}{143} & 0 & \frac{14}{143} \\ 0 & 1 & 0 & -\frac{2}{13} & \frac{47}{143} & -\frac{17}{143} & 0 & \frac{8}{143} \\ 0 & 0 & 1 & 0 & -\frac{1}{11} & \frac{2}{11} & 0 & \frac{1}{11} \\ \hline 0 & 0 & 0 & \frac{1}{13} & \frac{9}{143} & \frac{15}{143} & 1 & \frac{35}{143} \end{array}\right] \end{array}$$

The optimal solution of the primal and dual problems, respectively, are

$\overline{y}_1 = \frac{14}{143}, \overline{y}_2 = \frac{8}{143}, \overline{y}_3 = \frac{1}{11}$, and $\overline{x}_1 = \frac{1}{13}, \overline{x}_2 = \frac{9}{143}, \overline{x}_3 = \frac{15}{143}$, with $\lambda = \frac{35}{143}$.

The corresponding optimal mixed strategies for the column and row players, respectively, are

$$\hat{\mathbf{y}} = \overline{\mathbf{y}}/\lambda = \overline{\mathbf{y}} \cdot \tfrac{143}{35} = \begin{bmatrix} -\frac{14}{35} \\ \frac{8}{35} \\ \frac{13}{35} \end{bmatrix} \text{ and } \hat{\mathbf{x}} = \overline{\mathbf{x}}/\lambda = \overline{\mathbf{x}} \cdot \tfrac{143}{35} = \begin{bmatrix} \frac{11}{35} \\ \frac{9}{35} \\ \frac{15}{35} \end{bmatrix}$$

The value of the game with the shifted payoff matrix is $1/\lambda$, which is $\frac{143}{35}$, so the value of original game is $\frac{143}{35} - 3 = \frac{38}{35}$. Using the optimal strategy $\hat{\mathbf{x}}$, Bob should invest $\frac{11}{35}$ of the \$35,000 in stocks, $\frac{9}{35}$ in bonds, and $\frac{15}{45}$ in gold. That is, Bob should invest \$11,000 in stocks, \$9,000 in bonds, and \$15,000 in gold. The expected value of the game is $\frac{38}{35}$, based on \$100 for each play of the game. (The payoff matrix lists the amounts gained or lost for each \$100 that is invested for one year.) With \$35,000 to invest, Bob "plays" this game 350 times. Thus, he should expect to gain \$380, and the expected value of his portfolio at the end of the year is \$35,380.

**28** **a.** Consider $\mathbf{x} \in \mathscr{F}$ and $\mathbf{y} \in \mathscr{F}^*$, and note that $f(\mathbf{x}) = \mathbf{c}^T\mathbf{x} = \mathbf{x}^T\mathbf{c}$, and $g(\mathbf{y}) = \mathbf{b}^T\mathbf{y} = \mathbf{y}^T\mathbf{b}$. Because the entries in $\mathbf{x}$ and $\mathbf{y}$ are nonnegative, the inequalities $\mathbf{c} \le A^T\mathbf{y}$ and $A\mathbf{x} \le \mathbf{b}$ lead to

$f(\mathbf{x}) = \mathbf{x}^T\mathbf{c} \le \mathbf{x}^T A^T\mathbf{y} = (A\mathbf{x})^T\mathbf{y} = \mathbf{y}^T(A\mathbf{x}) \le \mathbf{y}^T(\mathbf{b}) = g(\mathbf{y})$

**b.** If $f(\hat{\mathbf{x}}) = g(\hat{\mathbf{y}})$, then for any $\mathbf{x} \in \mathscr{F}$, part (a) shows that $f(\mathbf{x}) \le g(\hat{\mathbf{y}}) = f(\hat{\mathbf{x}})$, so $\hat{\mathbf{x}}$ is an optimal solution to $P$. Similarly, for any $\mathbf{y} \in \mathscr{F}^*$, $g(\mathbf{y}) \ge f(\hat{\mathbf{x}}) = g(\hat{\mathbf{y}})$, which shows that $\hat{\mathbf{y}}$ is an optimal solution to $P^*$.

**29.** **a.** The coordinates of $\overline{\mathbf{x}}$ are all nonnegative. From the definition of $\mathbf{u}$, $\lambda$ is equal to the sum of these coordinates. It follows that the coordinates of $\hat{\mathbf{x}}$ are nonnegative and sum to one. Thus, $\hat{\mathbf{x}}$ is a mixed strategy for the row player $R$. A similar argument holds for $\hat{\mathbf{y}}$ and the column player $C$.

**b.** If $\mathbf{y}$ is any mixed strategy for $C$, then

$$E(\hat{\mathbf{x}}, \mathbf{y}) = \hat{\mathbf{x}}^T A\mathbf{y} = \frac{1}{\lambda}\left(\overline{\mathbf{x}}^T A\mathbf{y}\right) = \frac{1}{\lambda}\left[\left(A^T\overline{\mathbf{x}}\right)\cdot \mathbf{y}\right] \ge \frac{1}{\lambda}(\mathbf{v}\cdot\mathbf{y}) = \frac{1}{\lambda}$$

**c.** If $\mathbf{x}$ is any mixed strategy for $R$, then

$$E(\mathbf{x}, \hat{\mathbf{y}}) = \mathbf{x}^T A\hat{\mathbf{y}} = \frac{1}{\lambda}\left(\mathbf{x}^T A\overline{\mathbf{y}}\right) = \frac{1}{\lambda}[\mathbf{x}\cdot A\overline{\mathbf{y}}] \le \frac{1}{\lambda}(\mathbf{x}\cdot\mathbf{u}) = \frac{1}{\lambda}$$

**d.** Part (b) implies $v(\hat{\mathbf{x}}) \ge 1/\lambda$, so $v_R \ge 1/\lambda$. Part (c) implies $v(\hat{\mathbf{y}}) \le 1/\lambda$, so $v_C \le 1/\lambda$. It follows from the Minimax Theorem in Section 9.1 that $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are optimal mixed strategies for $R$ and $C$, respectively, and that the value of the game is $1/\lambda$.

# Chapter 9 - Supplementary Exercises

**1.** True. When $R$ "wins" a negative amount, say $-k$, it is the same as $R$ losing $k$.

**2.** False. See Example 2 in Section 9.1

**3.** False. The entries must also be nonnegative.

**4.** False. If **x** is a pure strategy, then one of the coordinates in **x** is a 1 and all the other coordinates are 0.

**5.** True. See footnote 2 in Section 9.1.

**6.** False. It should be $\mathbb{R}^m$, not $\mathbb{R}^n$.

**7.** True**.** By the definition of an optimal strategy.

**8.** True. See the comment following equation (2) in Section 9.1.

**9.** True. Theorem 4.

**10.** False. It is the minimum of $n$ linear functions of $t$. See the comment following equation (4) in Section 9.1.

**11.** True**.** Theorem 6.

**12.** False**.** See Example 5 in Section 9.2.

**13.** False**.** The problem is unbounded, not infeasible. If it were infeasible, then the feasible set would be empty.

**14.** True**.** If the problem is unbounded, then the feasible set is nonempty and the problem is feasible.

**15.** False**.** If the *objective function* is unbounded on the feasible set, then there is no optimal solution. But the feasible set may be unbounded and the problem still have an optimal solution (depending on the coefficients in the objective function). See Exercise 25 in these supplementary exercises for an example.

**16.** True**.** See the comment before the definition of a slack variable.

**17.** False**.** Matrix $A$ has $m$ rows, so $A\mathbf{x} \le \mathbf{b}$ has $m$ inequalities, and $m$ slack variables are required.

**18.** False. In Example 6 in Section 9.3, the slack variable $x_4$ is in the initial basic feasible solution, with $x_4 = 20$. In the next tableau it is out of the solution, with $x_4 = 0$. In the final tableau, it is back in the solution, with $x_4 = 8$.

**19.** True**.** See Example 7 in Section 9.3.

**20.** False**.** The negative of the coefficients of the objective function go in the bottom row.

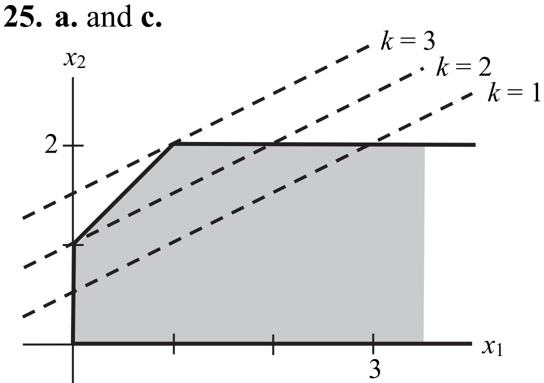
**21.** False**.** The matrix $A$ is replaced by its transpose, $A^T$.

**22.** True. If the primal program has an optimal solution, then the dual program will also have an optimal solution (Theorem 7), so the dual program must be bounded.

**23.** True. See Theorem 7.

**24.** False. The value of the game is $1/\lambda$.

**25. a.** and **c.**




$f(x_1, x_2) = -x_1 + 2x_2 = k$

**b.** The extreme points are (0, 0), (0, 1), and (1, 2).

| Extreme Point | $-x_1 + 2x_2$ |
|---|---|
| (0, 0) | 0 |
| (0, 1) | 2 |
| (1, 2) | 3 |

max.

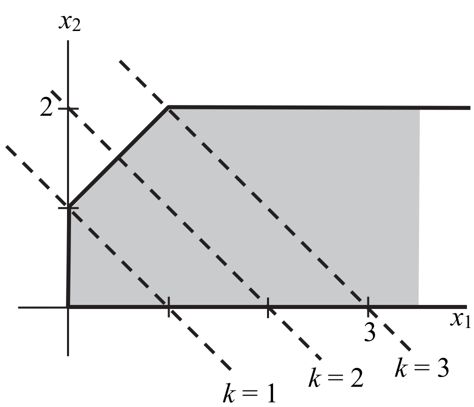
**d.** The optimal solution is $x_1 = 1$ and $x_2 = 2$ with $f(x_1, x_2) = 3$.

**26.**

$$\begin{array}{c} x_1 \quad x_2 \quad x_3 \quad x_4 \quad M \\ \left[\begin{array}{rrrrr|r} -1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 2 \\ \hline 1 & -2 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \rightarrow \begin{array}{c} x_1 \quad x_2 \quad x_3 \quad x_4 \quad M \\ \left[\begin{array}{rrrrr|r} -1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 1 & 0 & 1 \\ \hline -1 & 0 & 2 & 0 & 1 & 2 \end{array}\right] \end{array} \rightarrow \begin{array}{c} x_1 \quad x_2 \quad x_3 \quad x_4 \quad M \\ \left[\begin{array}{rrrrr|r} 0 & 1 & 0 & 1 & 0 & 2 \\ 1 & 0 & -1 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 1 & 1 & 3 \end{array}\right] \end{array}$$

$x_1 = 0,\ x_2 = 0,\ M = 0$ $\qquad$ $x_1 = 0,\ x_2 = 1,\ M = 2$ $\qquad$ $x_1 = 1,\ x_2 = 2,\ M = 3$

**27. a.**




$f(x_1, x_2) = x_1 + x_2 = k$

**b.** The extreme points are (0, 0), (0, 1), and (1, 2).

| Extreme Point | $x_1 + x_2$ |
|---|---|
| (0, 0) | 0 |
| (0, 1) | 1 |
| (1, 2) | 3 |

**28.**

$$\begin{array}{c} x_1 \quad x_2 \quad x_3 \quad x_4 \quad M \\ \left[\begin{array}{rrrrr|r} -1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 2 \\ \hline -1 & -1 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \rightarrow \begin{array}{c} x_1 \quad x_2 \quad x_3 \quad x_4 \quad M \\ \left[\begin{array}{rrrrr|r} -1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & -1 & 1 & 0 & 1 \\ \hline -2 & 0 & 1 & 0 & 1 & 1 \end{array}\right] \end{array} \rightarrow \begin{array}{c} x_1 \quad x_2 \quad x_3 \quad x_4 \quad M \\ \left[\begin{array}{rrrrr|r} 0 & 1 & 0 & 1 & 0 & 2 \\ 1 & 0 & -1 & 1 & 0 & 1 \\ \hline 0 & 0 & -1 & 2 & 1 & 3 \end{array}\right] \end{array}$$

$x_1 = 0,\ x_2 = 0,\ M = 0$ $\qquad$ $x_1 = 0,\ x_2 = 1,\ M = 2$ $\qquad$ $x_1 = 1,\ x_2 = 2,\ M = 3$

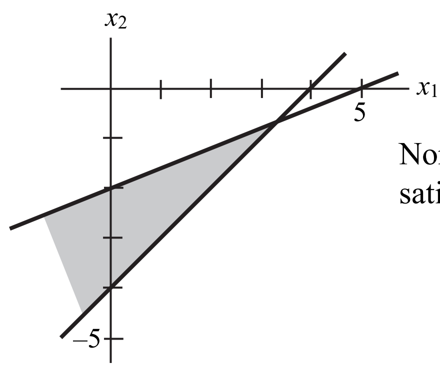
The –1 in the bottom row and third column of the last matrix tells us to bring $x_3$ into the solution. But there is no positive entry in the third column to use as a pivot. This indicates the objective function is unbounded.

**29. a.** and **b.**

$$\begin{array}{c}\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & M & \end{array}\\ \left[\begin{array}{ccccc|c} 1 & -1 & 1 & 0 & 0 & 4 \\ -2 & 5 & 0 & 1 & 0 & -10 \\ \hline -3 & -4 & 0 & 0 & 1 & 0 \end{array}\right]\end{array} \rightarrow \begin{array}{c}\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & M & \end{array}\\ \left[\begin{array}{ccccc|c} 1 & -1 & 1 & 0 & 0 & 4 \\ 0 & 3 & 2 & 1 & 0 & -2 \\ \hline 0 & -7 & 3 & 0 & 1 & 12 \end{array}\right]\end{array}$$

To get a nonnegative entry in the second row and sixth column, we bring $x_1$ into the solution. (It's entry in the second row is also negative.) We pivot on the 1 in the first row since $4 \div 1 = 4$, $(-10) \div (-2) = 5$, and $4 < 5$. The –10 has increased to –2 in the second tableau, but it is still negative. Unfortunately, there is now no other negative entry in the second row to identify another variable to bring into the "solution." That means there is no way to eliminate the –2, and this tells us there are no feasible solutions.

**c.**




None of the points in the shaded area satisfy $x_2 \geq 0$, so the feasible set is empty.

**30. a.** Here are the simplex tableaus.

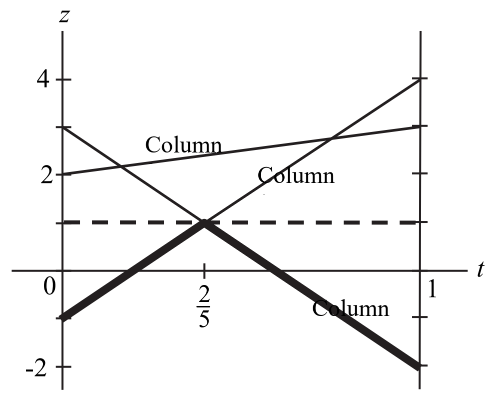
$$\begin{array}{c}\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M & \end{array}\\ \left[\begin{array}{ccccccc|c} 1 & 2 & -1 & 1 & 0 & 0 & 0 & 16 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 12 \\ 2 & 2 & 1 & 0 & 0 & 1 & 0 & 36 \\ \hline -4 & -5 & 1 & 0 & 0 & 0 & 1 & 0 \end{array}\right]\end{array} \Rightarrow \begin{array}{c}\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M & \end{array}\\ \left[\begin{array}{ccccccc|c} \frac{1}{2} & 1 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 8 \\ \frac{1}{2} & 0 & \frac{1}{2} & -\frac{1}{2} & 1 & 0 & 0 & 4 \\ 1 & 0 & 2 & -1 & 0 & 1 & 0 & 20 \\ \hline -\frac{3}{2} & 0 & -\frac{3}{2} & \frac{5}{2} & 0 & 0 & 1 & 40 \end{array}\right]\end{array} \Rightarrow$$

$$\begin{array}{c}\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M & \end{array}\\ \left[\begin{array}{ccccccc|c} 0 & 1 & -1 & 1 & -1 & 0 & 0 & 4 \\ 1 & 0 & 1 & -1 & 2 & 0 & 0 & 8 \\ 0 & 0 & 1 & 0 & -2 & 1 & 0 & 12 \\ \hline 0 & 0 & 0 & 1 & 3 & 0 & 1 & 52 \end{array}\right]\end{array} \Rightarrow \begin{array}{c}\begin{array}{ccccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & M & \end{array}\\ \left[\begin{array}{ccccccc|c} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 12 \\ 1 & 0 & 1 & -1 & 2 & 0 & 0 & 8 \\ -1 & 0 & 0 & 1 & -4 & 1 & 0 & 4 \\ \hline 0 & 0 & 0 & 1 & 3 & 0 & 1 & 52 \end{array}\right]\end{array}$$

$x_1 = 8,\ x_2 = 4,\ x_3 = 0,\ M = 52$ $\qquad$ $x_1 = 0,\ x_2 = 12,\ x_3 = 8.\ M = 52$

**b.** The feasible set $\mathcal{F}$ is a polyhedron in $\mathbb{R}^3$. The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are extreme points of $\mathcal{F}$. Let $H$ be the level plane $\{\mathbf{x} \in \mathbb{R}^3 : f(\mathbf{x}) = 52\}$, where $f(\mathbf{x})$ is the objective function. Then $H \cap \mathcal{F}$ is the line segment from $\mathbf{v}_1$ to $\mathbf{v}_2$. Any point on this line segment will correspond to an optimal solution of the problem.

**31. a.** We begin by graphing the three lines $z = a_{1j}(1-t) + a_{2j}t$ for $0 \le t \le 1$ on a $t$-$z$ coordinate system. The graph of the heavy bent line is $z = v(\mathbf{x}(t))$. Its maximum is seen to occur at the intersection of the lines for Column 1 and Column 3.




$$\left.\begin{array}{l} \text{Col. 1: } z = -1 + 5t \\ \text{Col. 3: } z = 3 - 5t \end{array}\right\} \Rightarrow t = \frac{2}{5} \text{ and } z =$$

So the value of the game $v$ is 1 and the optimal strategy for $R$ is $\hat{\mathbf{x}} = \begin{bmatrix} 1-t \\ t \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{2}{5} \end{bmatrix}$. To find the optimal strategy for $C$, we let $\hat{\mathbf{y}} = \begin{bmatrix} c_1 \\ 0 \\ 1-c_1 \end{bmatrix}$.

Then $v = 1 = E(\mathbf{e}_1, \hat{\mathbf{y}}) = (-1)(c_1) + (2)(0) + (3)(1-c_1) \Rightarrow c_1 = \frac{1}{2}$. So $\hat{\mathbf{y}} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$.

As a check we may compute $E(\mathbf{e}_2, \hat{\mathbf{y}}) = (4)(\frac{1}{2}) + (3)(0) + (-2)(\frac{1}{2}) = 1 = v$.

**32.** To obtain a matrix $B$ with positive entries, add 3 to each entry in $A$:

$$B = \begin{bmatrix} -1 & 2 & 3 \\ 4 & 3 & -2 \end{bmatrix} + \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 6 \\ 7 & 6 & 1 \end{bmatrix}$$

The optimal strategy for the column player $C$ is found by solving the linear programing problem

$$\begin{array}{ll} \text{Maximize} & y_1 + y_2 + y_3 \\ \text{subject to} & 2y_1 + 5y_2 + 6y_3 \le 1 \\ & 7y_1 + 6y_2 + y_3 \le 1 \end{array}$$

and $y_1 \ge 0, y_2 \ge 0, y_3 \ge 0.$

$$\begin{array}{c} \begin{array}{cccccc} y_1 & y_2 & y_3 & y_4 & y_5 & M \end{array} \\ \left[\begin{array}{cccccc|c} 2 & 5 & 6 & 1 & 0 & 0 & 1 \\ 7 & 6 & 1 & 0 & 1 & 0 & 1 \\ \hline -1 & -1 & -1 & 0 & 0 & 1 & 0 \end{array}\right] \end{array} \Rightarrow \begin{array}{c} \begin{array}{cccccc} y_1 & y_2 & y_3 & y_4 & y_5 & M \end{array} \\ \left[\begin{array}{cccccc|c} 0 & \frac{23}{7} & \frac{40}{7} & 1 & -\frac{2}{7} & 0 & \frac{5}{7} \\ 1 & \frac{6}{7} & \frac{1}{7} & 0 & \frac{1}{7} & 0 & \frac{1}{7} \\ \hline 0 & -\frac{1}{7} & -\frac{6}{7} & 0 & \frac{1}{7} & 1 & \frac{1}{7} \end{array}\right] \end{array} \Rightarrow$$

$$\begin{array}{c} \begin{array}{cccccc} y_1 & y_2 & y_3 & y_4 & y_5 & M \end{array} \\ \left[\begin{array}{cccccc|c} 0 & \frac{23}{40} & 1 & \frac{7}{40} & -\frac{1}{20} & 0 & \frac{1}{8} \\ 1 & \frac{31}{40} & 0 & -\frac{1}{40} & \frac{3}{20} & 0 & \frac{1}{8} \\ \hline 0 & \frac{7}{20} & 0 & \frac{3}{20} & \frac{1}{10} & 1 & \frac{1}{4} \end{array}\right] \end{array}$$

The optimal solution of the primal problem for payoff matrix $B$ is $\bar{y}_1 = \frac{1}{8}$, $\bar{y}_2 = 0$, $\bar{y}_3 = \frac{1}{8}$, with $\lambda = \bar{y}_1 + \bar{y}_2 + \bar{y}_3 = \frac{1}{4}$. The corresponding optimal mixed strategy for the column player $C$ is $\hat{\mathbf{y}} = \bar{\mathbf{y}}/\lambda = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$. The optimal solution of the dual problem comes from the bottom entries under the slack variables: $\bar{x}_1 = \frac{3}{20}$ and $\bar{x}_2 = \frac{1}{10}$, with $\lambda = \bar{x}_1 + \bar{x}_2 = \frac{1}{4}$

Thus the optimal mixed strategy for the row player $R$ is $\hat{\mathbf{x}} = \bar{\mathbf{x}}/\lambda = \begin{bmatrix} \frac{3}{5} \\ \frac{2}{5} \end{bmatrix}$.

The value of the game with payoff matrix $B$ is $v = \frac{1}{\lambda} = 4$, so the value of the original matrix game $A$ is $4 - 3 = 1$.