4 Vector Spaces

4.1 - Vector Spaces and Subspaces

Notes: This section is designed to avoid the standard exercises in which a student is asked to check ten axioms on an array of sets. Theorem 1 provides the main homework tool in this section for showing that a set is a subspace. Students should be taught how to check the closure axioms. The exercises in this section (and the next few sections) emphasize $\mathbb{R}^n$, to give students time to absorb the abstract concepts. Other vectors do appear later in the chapter: the space $\mathbb{S}$ of signals is used in Sections 4.7 and 4.8, and the spaces $\mathbb{P}_n$ of polynomials are used in many sections of Chapters 4 and 6.

1. **a**. If **u** and **v** are in V, then their entries are nonnegative. Since a sum of nonnegative numbers is nonnegative, the vector **u** + **v** has nonnegative entries. Thus **u** + **v** is in V.

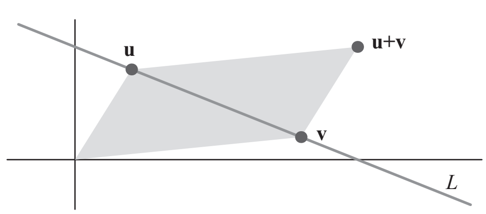
b. *Example:* If $\mathbf{u} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $c = -1$, then **u** is in V but c**u** is not in V.

2. **a**. If $\mathbf{u} = \begin{bmatrix} x \\ y \end{bmatrix}$ is in W, then the vector $c\mathbf{u} = c\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix}$ is in W because $(cx)(cy) = c^2(xy) \geq 0$ since $xy \geq 0$.

b. *Example:* If $\mathbf{u} = \begin{bmatrix} -1 \\ -7 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, then **u** and **v** are in W but **u** + **v** is not in W.

3. *Example:* If $\mathbf{u} = \begin{bmatrix} .5 \\ .5 \end{bmatrix}$ and $c = 4$, then **u** is in H but c**u** is not in H. Since H is not closed under scalar multiplication, H is not a subspace of $\mathbb{R}^2$.

4. Note that **u** and **v** are on the line L, but **u** + **v** is not.



5. Yes. Since the set is $\text{Span}\{t^2\}$, the set is a subspace by Theorem 1.

6. No. The zero vector is not in the set.

7. No. The set is not closed under multiplication by scalars which are not integers.

8. Yes. The zero vector is in the set H. If $\mathbf{p}$ and $\mathbf{q}$ are in H, then $(\mathbf{p}+\mathbf{q})(0) = \mathbf{p}(0) + \mathbf{q}(0) = 0 + 0 = 0$, so $\mathbf{p}+\mathbf{q}$ is in H. For any scalar c, $(c\mathbf{p})(0) = c \cdot \mathbf{p}(0) = c \cdot 0 = 0$, so $c\mathbf{p}$ is in H. Thus H is a subspace by Theorem 1.

9. The set $H = \text{Span}\{\mathbf{v}\}$, where $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$. Thus H is a subspace of $\mathbb{R}^3$ by Theorem 1.

10. The set $H = \text{Span}\{\mathbf{v}\}$, where $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$. Thus H is a subspace of $\mathbb{R}^3$ by Theorem 1.

11. The set $W = \text{Span}\{\mathbf{u}, \mathbf{v}\}$, where $\mathbf{u} = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. Thus W is a subspace of $\mathbb{R}^3$ by Theorem 1.

12. The set $W = \text{Span}\{\mathbf{u}, \mathbf{v}\}$, where $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix}$. Thus W is a subspace of $\mathbb{R}^4$ by Theorem 1.

13. **a**. The vector $\mathbf{w}$ is not in the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. There are 3 vectors in the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

b. The set $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ contains infinitely many vectors.

c. The vector $\mathbf{w}$ is in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if and only if the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{w}$ has a solution. Row reducing the augmented matrix for this system of linear equations gives

$$\begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 2 & 1 \\ -1 & 3 & 6 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

so the equation has a solution and $\mathbf{w}$ is in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

14. The augmented matrix is found as in Exercise 13c. Since $\begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 1 & 2 & 4 \\ -1 & 3 & 6 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$,

the equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{w}$ has no solution, and $\mathbf{w}$ is not in the subspace spanned by $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

15. Since the zero vector is not in W, W is not a vector space.

16. Since the zero vector is not in W, W is not a vector space.

17. Since a vector $\mathbf{w}$ in W may be written as $\mathbf{w} = a\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + c\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$, $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a set that spans W.

18. Since a vector $\mathbf{w}$ in W may be written as $\mathbf{w} = a\begin{bmatrix} 4 \\ 0 \\ 1 \\ -2 \end{bmatrix} + b\begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $S = \left\{ \begin{bmatrix} 4 \\ 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a set that spans W.

19. Let H be the set of all functions described by $y(t) = c_1 \cos \omega t + c_2 \sin \omega t$. Then H is a subset of the vector space V of all real-valued functions, and may be written as $H = \text{Span}\,\{\cos \omega t, \sin \omega t\}$. By Theorem 1, H is a subspace of V and is hence a vector space.

20. **a**. The following facts about continuous functions must be shown.

1. The constant function $\mathbf{f}(t) = 0$ is continuous.
2. The sum of two continuous functions is continuous.
3. A constant multiple of a continuous function is continuous.

b. Let $H = \{\mathbf{f} \text{ in } C[a, b]: \mathbf{f}(a) = \mathbf{f}(b)\}$.

1. Let $\mathbf{g}(t) = 0$ for all t in $[a, b]$. Then $\mathbf{g}(a) = \mathbf{g}(b) = 0$, so $\mathbf{g}$ is in H.
2. Let $\mathbf{g}$ and $\mathbf{h}$ be in H. Then $\mathbf{g}(a) = \mathbf{g}(b)$ and $\mathbf{h}(a) = \mathbf{h}(b)$, and $(\mathbf{g} + \mathbf{h})(a) = \mathbf{g}(a) + \mathbf{h}(a) = \mathbf{g}(b) + \mathbf{h}(b) = (\mathbf{g} + \mathbf{h})(b)$, so $\mathbf{g} + \mathbf{h}$ is in H.
3. Let $\mathbf{g}$ be in H. Then $\mathbf{g}(a) = \mathbf{g}(b)$, and $(c\mathbf{g})(a) = c\mathbf{g}(a) = c\mathbf{g}(b) = (c\mathbf{g})(b)$, so $c\mathbf{g}$ is in H.

Thus H is a subspace of $C[a, b]$.

21. The set H is a subspace of $M_{2\times 2}$. The zero matrix is in H, the sum of two upper triangular matrices is upper triangular, and a scalar multiple of an upper triangular matrix is upper triangular.

22. The set H is a subspace of $M_{2\times 4}$. The 2×4 zero matrix 0 is in H because $F0 = 0$. If A and B are matrices in H, then $F(A + B) = FA + FB = 0 + 0 = 0$, so $A + B$ is in H. If A is in H and c is a scalar, then $F(cA) = c(FA) = c0 = 0$, so cA is in H.

23. False. The zero vector in V is the function $\mathbf{f}$ whose values $\mathbf{f}(t)$ are zero *for all t* in $\mathbb{R}$.

24. True. See the definition of a vector space.

25. True. See Example 2.

26. True. See statement (3) in the box before Example 1.

27. False. See Exercises 1, 2, and 3 for examples of subsets which contain the zero vector but are not subspaces.

28. True. See the paragraph before Example 6.

29. True. See the paragraph before Example 6.

30. False. See Example 8.

31. True. By Example 4 and the definition of a subspace.

32. False. The second and third parts of the conditions are stated incorrectly. For example, part (ii) does not state that **u** and **v** represent all possible elements of *H*

33. 2, 4

34. **a**. 3 **b**. 5 **c**. 4

35. **a**. 8 **b**. 3 **c**. 5 **d**. 4

36. **a**. 4 **b**. 7 **c**. 3 **d**. 5 **e**. 4

37. Consider $\mathbf{u} + (-1)\mathbf{u}$. By Axiom 10, $\mathbf{u} + (-1)\mathbf{u} = 1\mathbf{u} + (-1)\mathbf{u}$. By Axiom 8, $1\mathbf{u} + (-1)\mathbf{u} = (1 + (-1))\mathbf{u} = 0\mathbf{u}$. By Exercise 35, $0\mathbf{u} = \mathbf{0}$. Thus $\mathbf{u} + (-1)\mathbf{u} = \mathbf{0}$, and by Exercise 34 $(-1)\mathbf{u} = -\mathbf{u}$.

38. By Axiom 10, $\mathbf{u} = 1\mathbf{u}$. Since c is nonzero, $c^{-1}c = 1$, and $\mathbf{u} = (c^{-1}c)\mathbf{u}$. By Axiom 9, $(c^{-1}c)\mathbf{u} = c^{-1}(c\mathbf{u}) = c^{-1}\mathbf{0}$ since $c\mathbf{u} = \mathbf{0}$. Thus $\mathbf{u} = c^{-1}\mathbf{0} = \mathbf{0}$ by Property (2), proven in Exercise 36.

39. Any subspace H that contains **u** and **v** must also contain all scalar multiples of **u** and **v**, and hence must also contain all sums of scalar multiples of **u** and **v**. Thus H must contain all linear combinations of **u** and **v**, or Span $\{\mathbf{u}, \mathbf{v}\}$.

Note: Exercises 40–42 provide good practice for mathematics majors because these arguments involve simple symbol manipulation typical of mathematical proofs. Most students outside mathematics might profit more from other types of exercises.

40. Both H and K contain the zero vector of V because they are subspaces of V. Thus the zero vector of V is in $H \cap K$. Let **u** and **v** be in $H \cap K$. Then **u** and **v** are in H. Since H is a subspace $\mathbf{u} + \mathbf{v}$ is in H. Likewise **u** and **v** are in K. Since K is a subspace $\mathbf{u} + \mathbf{v}$ is in K. Thus $\mathbf{u} + \mathbf{v}$ is in $H \cap K$. Let **u** be in $H \cap K$. Then **u** is in H. Since H is a subspace $c\mathbf{u}$ is in H. Likewise **v** is in K. Since K is a subspace $c\mathbf{u}$ is in K. Thus $c\mathbf{u}$ is in $H \cap K$ for any scalar c, and $H \cap K$ is a subspace of V.

The union of two subspaces is not in general a subspace. For an example in $\mathbb{R}^2$ let H be the x-axis and let K be the y-axis. Then both H and K are subspaces of $\mathbb{R}^2$, but $H \cup K$ is not closed under vector addition. The subset $H \cup K$ is thus not a subspace of $\mathbb{R}^2$.

41. **a**. Given subspaces H and K of a vector space V, the zero vector of V belongs to $H + K$, because $\mathbf{0}$ is in both H and K (since they are subspaces) and $\mathbf{0} = \mathbf{0} + \mathbf{0}$. Next, take two vectors in $H + K$, say $\mathbf{w}_1 = \mathbf{u}_1 + \mathbf{v}_1$ and $\mathbf{w}_2 = \mathbf{u}_2 + \mathbf{v}_2$ where $\mathbf{u}_1$ and $\mathbf{u}_2$ are in H, and $\mathbf{v}_1$ and $\mathbf{v}_2$ are in K. Then

$$\mathbf{w}_1 + \mathbf{w}_2 = (\mathbf{u}_1 + \mathbf{v}_1) + (\mathbf{u}_2 + \mathbf{v}_2) = (\mathbf{u}_1 + \mathbf{u}_2) + (\mathbf{v}_1 + \mathbf{v}_2)$$

because vector addition in V is commutative and associative. Now $\mathbf{u}_1 + \mathbf{u}_2$ is in H and $\mathbf{v}_1 + \mathbf{v}_2$ is in K because H and K are subspaces. This shows that $\mathbf{w}_1 + \mathbf{w}_2$ is in $H + K$. Thus $H + K$ is closed under addition of vectors. Finally, for any scalar c,

$$c\mathbf{w}_1 = c(\mathbf{u}_1 + \mathbf{v}_1) = c\mathbf{u}_1 + c\mathbf{v}_1$$

The vector $c\mathbf{u}_1$ belongs to H and $c\mathbf{v}_1$ belongs to K, because H and K are subspaces. Thus, $c\mathbf{w}_1$ belongs to $H + K$, so $H + K$ is closed under multiplication by scalars. These arguments show that $H + K$ satisfies all three conditions necessary to be a subspace of V.

b. Certainly H is a subset of $H + K$ because every vector $\mathbf{u}$ in H may be written as $\mathbf{u} + \mathbf{0}$, where the zero vector $\mathbf{0}$ is in K (and also in H, of course). Since H contains the zero vector of $H + K$, and H is closed under vector addition and multiplication by scalars (because H is a subspace of V), H is a subspace of $H + K$. The same argument applies when H is replaced by K, so K is also a subspace of $H + K$.

42. A proof that $H + K = \text{Span}\{\mathbf{u}_1,\ldots,\mathbf{u}_p,\mathbf{v}_1,\ldots,\mathbf{v}_q\}$ has two parts. First, one must show that $H + K$ is a subset of $\text{Span}\{\mathbf{u}_1,\ldots,\mathbf{u}_p,\mathbf{v}_1,\ldots,\mathbf{v}_q\}$. Second, one must show that $\text{Span}\{\mathbf{u}_1,\ldots,\mathbf{u}_p,\mathbf{v}_1,\ldots,\mathbf{v}_q\}$ is a subset of $H + K$.

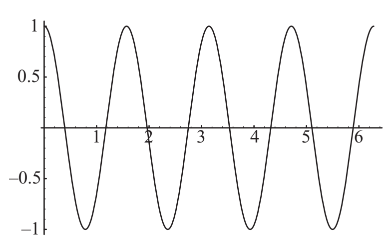
(1) A typical vector H has the form $c_1\mathbf{u}_1 + \ldots + c_p\mathbf{u}_p$ and a typical vector in K has the form $d_1\mathbf{v}_1 + \ldots + d_q\mathbf{v}_q$. The sum of these two vectors is a linear combination of $\mathbf{u}_1,\ldots,\mathbf{u}_p,\mathbf{v}_1,\ldots,\mathbf{v}_q$ and so belongs to $\text{Span}\{\mathbf{u}_1,\ldots,\mathbf{u}_p,\mathbf{v}_1,\ldots,\mathbf{v}_q\}$. Thus $H + K$ is a subset of $\text{Span}\{\mathbf{u}_1,\ldots,\mathbf{u}_p,\mathbf{v}_1,\ldots,\mathbf{v}_q\}$.

(2) Each of the vectors $\mathbf{u}_1,\ldots,\mathbf{u}_p,\mathbf{v}_1,\ldots,\mathbf{v}_q$ belongs to $H + K$, by Exercise 41(b), and so any linear combination of these vectors belongs to $H + K$, since $H + K$ is a subspace, by Exercise 41(a). Thus, $\text{Span}\{\mathbf{u}_1,\ldots,\mathbf{u}_p,\mathbf{v}_1,\ldots,\mathbf{v}_q\}$ is a subset of $H + K$.

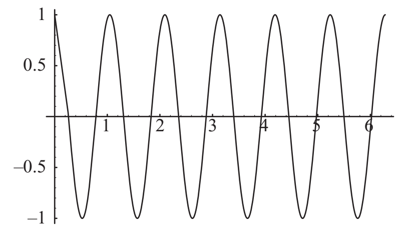
43. Since $\begin{bmatrix} 8 & -4 & -7 & 9 \\ -4 & 3 & 6 & -4 \\ -3 & -2 & -5 & -4 \\ 9 & -8 & -18 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\mathbf{w}$ is in the subspace spanned by $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\}$.

44. Since $\begin{bmatrix} A & \mathbf{y} \end{bmatrix} = \begin{bmatrix} 3 & -5 & -9 & -4 \\ 8 & 7 & -6 & -8 \\ -5 & -8 & 3 & 6 \\ 2 & -2 & -9 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1/5 \\ 0 & 1 & 0 & -2/5 \\ 0 & 0 & 1 & 3/5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\mathbf{y}$ is in the subspace spanned by the columns of A.

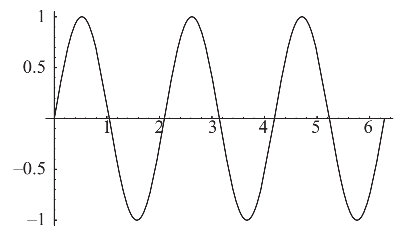
45. The graph of $\mathbf{f}(t)$ is given below. A conjecture is that $\mathbf{f}(t) = \cos 4t$.



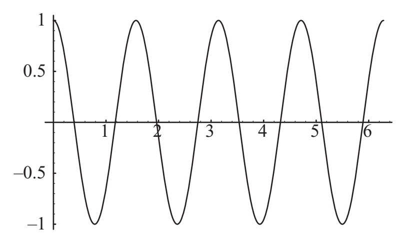
The graph of $\mathbf{g}(t)$ is given below. A conjecture is that $\mathbf{g}(t) = \cos 6t$.



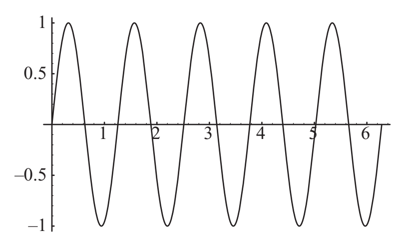
46. The graph of $\mathbf{f}(t)$ is given below. A conjecture is that $\mathbf{f}(t) = \sin 3t$.



The graph of $\mathbf{g}(t)$ is given below. A conjecture is that $\mathbf{g}(t) = \cos 4t$.



The graph of $\mathbf{h}(t)$ is given below. A conjecture is that $\mathbf{h}(t) = \sin 5t$.



4.2 - Null Spaces, Column Spaces, Row Spaces, and Linear Transformations

Notes: This section provides a review of Chapter 1 using the new terminology. Linear tranformations are introduced quickly since students are already comfortable with the idea from $\mathbb{R}^n$. The key exercises are 17–24, which are straightforward but help to solidify the notions of null spaces and column spaces. Exercises 42–48 deal with the kernel and range of a linear transformation and are progressively more advanced theoretically. The idea in Exercises 7–14 is for the student to use Theorems 1, 2, or 3 to determine whether a given set is a subspace.

1. One calculates that $A\mathbf{w} = \begin{bmatrix} 3 & -5 & -3 \\ 6 & -2 & 0 \\ -8 & 4 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, so **w** is in Nul A.

2. One calculates that $A\mathbf{w} = \begin{bmatrix} 5 & 21 & 19 \\ 13 & 23 & 2 \\ 8 & 14 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, so **w** is in Nul A.

3. First find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables. Since $\begin{bmatrix} A & \mathbf{0} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -7 & 6 & 0 \\ 0 & 1 & 4 & -2 & 0 \end{bmatrix}$, the general solution is $x_1 = 7x_3 - 6x_4$, $x_2 = -4x_3 + 2x_4$, with x_3 and x_4 free. So $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3\begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix}$, and a spanning set for Nul A is $\left\{ \begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

4. First find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables. Since $\begin{bmatrix} A & \mathbf{0} \end{bmatrix} \sim \begin{bmatrix} 1 & -6 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$, the general solution is $x_1 = 6x_2$, $x_3 = 0$, with x_2 and x_4 free. So $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2\begin{bmatrix} 6 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, and a spanning set for Nul A is $\left\{ \begin{bmatrix} 6 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

5. First find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables. Since

$[A \quad \mathbf{0}] \sim \begin{bmatrix} 1 & -2 & 0 & 4 & 0 & 0 \\ 0 & 0 & 1 & -9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$, the general solution is $x_1 = 2x_2 - 4x_4$, $x_3 = 9x_4$, $x_5 = 0$, with x_2 and x_4 free. So $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 9 \\ 1 \\ 0 \end{bmatrix}$, and a spanning set for Nul A is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 9 \\ 1 \\ 0 \end{bmatrix} \right\}$.

6. First find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables. Since $[A \quad \mathbf{0}] \sim \begin{bmatrix} 1 & 0 & 6 & -8 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, the general solution is $x_1 = -6x_3 + 8x_4 - x_5$, $x_2 = 2x_3 - x_4$, with x_3, x_4, and x_5 free. So $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, and a spanning set for Nul A is $\left\{ \begin{bmatrix} -6 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 8 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

7. The set W is a subset of $\mathbb{R}^3$. If W were a vector space (under the standard operations in $\mathbb{R}^3$), then it would be a subspace of $\mathbb{R}^3$. But W is not a subspace of $\mathbb{R}^3$ since the zero vector is not in W. Thus W is not a vector space.

8. The set W is a subset of $\mathbb{R}^3$. If W were a vector space (under the standard operations in $\mathbb{R}^3$), then it would be a subspace of $\mathbb{R}^3$. But W is not a subspace of $\mathbb{R}^3$ since the zero vector is not in W. Thus W is not a vector space.

9. The set W is the set of all solutions to the homogeneous system of equations $a - 2b - 4c = 0$, $2a - c - 3d = 0$. Thus $W = \text{Nul } A$, where $A = \begin{bmatrix} 1 & -2 & -4 & 0 \\ 2 & 0 & -1 & -3 \end{bmatrix}$. Thus W is a subspace of $\mathbb{R}^4$ by Theorem 2, and is a vector space.

10. The set W is the set of all solutions to the homogeneous system of equations $a + 3b - c = 0$, $a + b + c - d = 0$. Thus $W = \text{Nul } A$, where $A = \begin{bmatrix} 1 & 3 & -1 & 0 \\ 1 & 1 & 1 & -1 \end{bmatrix}$. Thus W is a subspace of $\mathbb{R}^4$ by Theorem 2, and is a vector space.

11. The set W is a subset of $\mathbb{R}^4$. If W were a vector space (under the standard operations in $\mathbb{R}^4$), then it would be a subspace of $\mathbb{R}^4$. But W is not a subspace of $\mathbb{R}^4$ since the zero vector is not in W. Thus W is not a vector space.

12. The set W is a subset of $\mathbb{R}^4$. If W were a vector space (under the standard operations in $\mathbb{R}^4$), then it would be a subspace of $\mathbb{R}^4$. But W is not a subspace of $\mathbb{R}^4$ since the zero vector is not in W. Thus W is not a vector space.

13. An element $\mathbf{w}$ in W may be written as $\mathbf{w} = c\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + d\begin{bmatrix} -6 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -6 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix}$, where c and d are any real numbers. So $W = \text{Col } A$ where $A = \begin{bmatrix} 1 & -6 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$. Thus W is a subspace of $\mathbb{R}^3$ by Theorem 3, and is a vector space.

14. An element $\mathbf{w}$ in W may be written as $\mathbf{w} = a\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix} + b\begin{bmatrix} 2 \\ -2 \\ -6 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -2 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix}$, where a and b are any real numbers. So $W = \text{Col } A$ where $A = \begin{bmatrix} -1 & 2 \\ 1 & -2 \\ 3 & -6 \end{bmatrix}$. Thus W is a subspace of $\mathbb{R}^3$ by Theorem 3, and is a vector space.

15. An element in this set may be written as $r\begin{bmatrix} 0 \\ 1 \\ 4 \\ 3 \end{bmatrix} + s\begin{bmatrix} 2 \\ 1 \\ 1 \\ -1 \end{bmatrix} + t\begin{bmatrix} 3 \\ -2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 1 & -2 \\ 4 & 1 & 0 \\ 3 & -1 & -1 \end{bmatrix}\begin{bmatrix} r \\ s \\ t \end{bmatrix}$, where r, s and t are any real numbers. So the set is Col A where $A = \begin{bmatrix} 0 & 2 & 3 \\ 1 & 1 & -2 \\ 4 & 1 & 0 \\ 3 & -1 & -1 \end{bmatrix}$.

16. An element in this set may be written as $b\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + c\begin{bmatrix} -1 \\ 1 \\ 5 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 1 \\ -4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 5 & -4 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} b \\ c \\ d \end{bmatrix}$, where b, c and d are any real numbers. So the set is Col A where $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 1 & 1 \\ 0 & 5 & -4 \\ 0 & 0 & 1 \end{bmatrix}$.

17. The matrix A is a 4×2 matrix. Thus

(a) Nul A is a subspace of $\mathbb{R}^2$, and

(b) Col A is a subspace of $\mathbb{R}^4$.

18. The matrix A is a 4×3 matrix. Thus

(a) Nul A is a subspace of $\mathbb{R}^3$, and

(b) Col A is a subspace of $\mathbb{R}^4$.

19. The matrix A is a 2×5 matrix. Thus

(a) Nul A is a subspace of $\mathbb{R}^5$, and

(b) Col A is a subspace of $\mathbb{R}^2$.

20. The matrix A is a 1×5 matrix. Thus

(a) Nul A is a subspace of $\mathbb{R}^5$, and

(b) Col A is a subspace of $\mathbb{R}^1 = \mathbb{R}$.

21. Either column of A is a nonzero vector in Col A. To find a nonzero vector in Nul A, find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables. Since $\begin{bmatrix} A & \mathbf{0} \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, the general solution is $x_1 = 3x_2$, with x_2 free. Letting x_2 be a nonzero value (say $x_2 = 1$) gives the nonzero vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, which is in Nul A. Any row of A is a nonzero vector in Row A.

22. Any column of A is a nonzero vector in Col A. To find a nonzero vector in Nul A, find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables. Since $\begin{bmatrix} A & \mathbf{0} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -7 & 6 & 0 \\ 0 & 1 & 4 & -2 & 0 \end{bmatrix}$, the general solution is $x_1 = 7x_3 - 6x_4$, $x_2 = -4x_3 + 2x_4$, with x_3 and x_4 free. Letting x_3 and x_4 be nonzero values (say $x_3 = x_4 = 1$) gives the nonzero vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \end{bmatrix}$, which is in Nul A. Any row of A is a nonzero vector in Row A.

23. Consider the system with augmented matrix $\begin{bmatrix} A & \mathbf{w} \end{bmatrix}$. Since $\begin{bmatrix} A & \mathbf{w} \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & -1/3 \\ 0 & 0 & 0 \end{bmatrix}$, the system is consistent and $\mathbf{w}$ is in Col A. Also, since $A\mathbf{w} = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbf{w}$ is in Nul A.

24. Consider the system with augmented matrix $\begin{bmatrix} A & \mathbf{w} \end{bmatrix}$. Since $\begin{bmatrix} A & \mathbf{w} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & -1/2 \\ 0 & 1 & 1/2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, the system is consistent, and $\mathbf{w}$ is in Col A. Also, since $A\mathbf{w} = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{w}$ is in Nul A.

25. True. See the definition before Example 1.

26. True. See Theorem 2.

27. False. See Theorem 2.

28. True. See Theorem 3.

29. True. See the remark just before Example 4.

30. False. See the box after Theorem 3.

31. False. The equation $A\mathbf{x} = \mathbf{b}$ must be consistent *for every* $\mathbf{b}$. See #7 in the table on page 206.

32. True. See the paragraph after the definition of a linear transformation.

33. True. See Figure 2.

34. True. See Figure 2.

35. True. See the remark after Theorem 3.

36. True. See the paragraph before Example 9.

37. True. See comment preceeding Example 5.

38. False. See the comment preceeding Example 5.

39. Let A be the coefficient matrix of the given homogeneous system of equations. Since $A\mathbf{x} = \mathbf{0}$ for $\mathbf{x} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{x}$ is in Nul$A$. Since Nul$A$ is a subspace of $\mathbb{R}^3$, it is closed under scalar multiplication.

Thus $10\mathbf{x} = \begin{bmatrix} 30 \\ 20 \\ -10 \end{bmatrix}$ is also in Nul$A$, and $x_1 = 30$, $x_2 = 20$, $x_3 = -10$ is also a solution to the system of equations.

40. Let A be the coefficient matrix of the given systems of equations. Since the first system has a solution, the constant vector $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 9 \end{bmatrix}$ is in Col$A$. Since Col A is a subspace of $\mathbb{R}^3$, it is closed under scalar multiplication. Thus $5\mathbf{b} = \begin{bmatrix} 0 \\ 5 \\ 45 \end{bmatrix}$ is also in Col A, and the second system of equations must thus have a solution.

41. **a**. Since $A\mathbf{0} = \mathbf{0}$, the zero vector is in Col A.

b. Since $A\mathbf{x} + A\mathbf{w} = A(\mathbf{x} + \mathbf{w}), A\mathbf{x} + A\mathbf{w}$ is in Col A.

c. Since $c(A\mathbf{x}) = A(c\mathbf{x}), cA\mathbf{x}$ is in Col A.

42. Since $T(\mathbf{0}_V) = \mathbf{0}_W$, the zero vector $\mathbf{0}_W$ of W is in the range of T. Let $T(\mathbf{x})$ and $T(\mathbf{w})$ be typical elements in the range of T. Then since $T(\mathbf{x}) + T(\mathbf{w}) = T(\mathbf{x} + \mathbf{w}), T(\mathbf{x}) + T(\mathbf{w})$ is in the range of T and the range of T is closed under vector addition. Let c be any scalar. Then since $cT(\mathbf{x}) = T(c\mathbf{x}), cT(\mathbf{x})$ is in the range of T and the range of T is closed under scalar multiplication. Hence the range of T is a subspace of W.

43. **a**. Let $\mathbf{p}$ and $\mathbf{q}$ be arbitary polynomials in $\mathbb{P}_2$, and let c be any scalar. Then

$$T(\mathbf{p} + \mathbf{q}) = \begin{bmatrix} (\mathbf{p} + \mathbf{q})(0) \\ (\mathbf{p} + \mathbf{q})(1) \end{bmatrix} = \begin{bmatrix} \mathbf{p}(0) + \mathbf{q}(0) \\ \mathbf{p}(1) + \mathbf{q}(1) \end{bmatrix} = \begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix} + \begin{bmatrix} \mathbf{q}(0) \\ \mathbf{q}(1) \end{bmatrix} = T(\mathbf{p}) + T(\mathbf{q})$$

and $T(c\mathbf{p}) = \begin{bmatrix} (c\mathbf{p})(0) \\ (c\mathbf{p})(1) \end{bmatrix} = c\begin{bmatrix} \mathbf{p}(0) \\ \mathbf{p}(1) \end{bmatrix} = cT(\mathbf{p})$, so T is a linear transformation.

b. Any quadratic polynomial $\mathbf{q}$ for which $\mathbf{q}(0) = 0$ and $\mathbf{q}(1) = 0$ will be in the kernel of T. The polynomial $\mathbf{q}$ must then be a multiple of $\mathbf{p}(t) = t(t-1)$. Given any vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ in $\mathbb{R}^2$, the polynomial $\mathbf{p} = x_1 + (x_2 - x_1)t$ has $\mathbf{p}(0) = x_1$ and $\mathbf{p}(1) = x_2$. Thus the range of T is all of $\mathbb{R}^2$.

44. Any quadratic polynomial $\mathbf{q}$ for which $\mathbf{q}(0) = 0$ will be in the kernel of T. The polynomial $\mathbf{q}$ must then be $\mathbf{q} = at + bt^2$. Thus the polynomials $\mathbf{p}_1(t) = t$ and $\mathbf{p}_2(t) = t^2$ span the kernel of T. If a vector is in the range of T, it must be of the form $\begin{bmatrix} a \\ a \end{bmatrix}$. If a vector is of this form, it is the image of the polynomial $\mathbf{p}(t) = a$ in $\mathbb{P}_2$. Thus the range of T is $\left\{ \begin{bmatrix} a \\ a \end{bmatrix} : a \text{ real} \right\}$.

45. **a**. For any A and B in $M_{2\times 2}$ and for any scalar c,

$$T(A + B) = (A + B) + (A + B)^T = A + B + A^T + B^T = (A + A^T) + (B + B^T) = T(A) + T(B)$$

and $T(cA) = (cA)^T = c(A^T) = cT(A)$, so T is a linear transformation.

b. Let B be an element of $M_{2\times 2}$ with $B^T = B$, and let $A = \frac{1}{2}B$. Then

$$T(A) = A + A^T = \frac{1}{2}B + (\frac{1}{2}B)^T = \frac{1}{2}B + \frac{1}{2}B^T = \frac{1}{2}B + \frac{1}{2}B = B$$

c. Part b. showed that the range of T contains the set of all B in $M_{2\times 2}$ with $B^T = B$. It must also be shown that any B in the range of T has this property. Let B be in the range of T. Then $B = T(A)$ for some A in $M_{2\times 2}$. Then $B = A + A^T$, and $B^T = (A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T = B$, so B has the property that $B^T = B$.

d. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be in the kernel of T. Then $T(A) = A + A^T = 0$, so

$$A + A^T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 2a & c+b \\ b+c & 2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Solving it is found that $a = d = 0$ and $c = -b$. Thus the kernel of T is

$\left\{ \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} : b \text{ real} \right\}$.

46. Let **f** and **g** be any elements in $C[0, 1]$ and let c be any scalar. Then $T(\mathbf{f})$ is the antiderivative **F** of **f** with $\mathbf{F}(0) = 0$ and $T(\mathbf{g})$ is the antiderivative **G** of **g** with $\mathbf{G}(0) = 0$. By the rules for antidifferentiation $\mathbf{F} + \mathbf{G}$ will be an antiderivative of $\mathbf{f} + \mathbf{g}$, and $(\mathbf{F} + \mathbf{G})(0) = \mathbf{F}(0) + \mathbf{G}(0) = 0 + 0 = 0.$ Thus $T(\mathbf{f} + \mathbf{g}) = T(\mathbf{f}) + T(\mathbf{g})$. Likewise $c\mathbf{F}$ will be an antiderivative of $c\mathbf{f}$, and $(c\mathbf{F})(0) = c\mathbf{F}(0) = c0 = 0$. Thus $T(c\mathbf{f}) = cT(\mathbf{f})$, and T is a linear transformation. To find the kernel of T, we must find all functions f in $C[0,1]$ with antiderivative equal to the zero function. The only function with this property is the zero function **0**, so the kernel of T is $\{\mathbf{0}\}$.

47. Since U is a subspace of V, $\mathbf{0}_V$ is in U. Since T is linear, $T(\mathbf{0}_V) = \mathbf{0}_W$. So $\mathbf{0}_W$ is in $T(U)$. Let $T(\mathbf{x})$ and $T(\mathbf{y})$ be typical elements in $T(U)$. Then **x** and **y** are in U, and since U is a subspace of V, $\mathbf{x} + \mathbf{y}$ is also in U. Since T is linear, $T(\mathbf{x}) + T(\mathbf{y}) = T(\mathbf{x} + \mathbf{y})$. So $T(\mathbf{x}) + T(\mathbf{y})$ is in $T(U)$, and $T(U)$ is closed under vector addition. Let c be any scalar. Then since **x** is in U and U is a subspace of V, $c\mathbf{x}$ is in U. Since T is linear, $T(c\mathbf{x}) = cT(\mathbf{x})$ and $cT(\mathbf{x})$ is in $T(U)$. Thus $T(U)$ is closed under scalar multiplication, and $T(U)$ is a subspace of W.

48. Since Z is a subspace of W, $\mathbf{0}_W$ is in Z. Since T is linear, $T(\mathbf{0}_V) = \mathbf{0}_W$. So $\mathbf{0}_V$ is in U. Let **x** and **y** be typical elements in U. Then $T(\mathbf{x})$ and $T(\mathbf{y})$ are in Z, and since Z is a subspace of W, $T(\mathbf{x}) + T(\mathbf{y})$ is also in Z. Since T is linear, $T(\mathbf{x}) + T(\mathbf{y}) = T(\mathbf{x} + \mathbf{y})$. So $T(\mathbf{x} + \mathbf{y})$ is in Z, and $\mathbf{x} + \mathbf{y}$ is in U. Thus U is closed under vector addition. Let c be any scalar. Then since **x** is in U, $T(\mathbf{x})$ is in Z. Since Z is a subspace of W, $cT(\mathbf{x})$ is also in Z. Since T is linear, $cT(\mathbf{x}) = T(c\mathbf{x})$ and $T(c\mathbf{x})$ is in $T(U)$. Thus $c\mathbf{x}$ is in U and U is closed under scalar multiplication. Hence U is a subspace of V.

49. Consider the system with augmented matrix $\begin{bmatrix} A & \mathbf{w} \end{bmatrix}$. Since

$$\begin{bmatrix} A & \mathbf{w} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1/95 & 1/95 \\ 0 & 1 & 0 & 39/19 & -20/19 \\ 0 & 0 & 1 & 267/95 & -172/95 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$, the system is consistent and $\mathbf{w}$ is in Col$A$. Also, since

$$A\mathbf{w} = \begin{bmatrix} 7 & 6 & -4 & 1 \\ -5 & -1 & 0 & -2 \\ 9 & -11 & 7 & -3 \\ 19 & -9 & 7 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -3 \end{bmatrix} = \begin{bmatrix} 14 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$, $\mathbf{w}$ is not in Nul$A$.

50. Consider the system with augmented matrix $\begin{bmatrix} A & \mathbf{w} \end{bmatrix}$. Since

$$\begin{bmatrix} A & \mathbf{w} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 & -2 \\ 0 & 1 & -2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$, the system is consistent and $\mathbf{w}$ is in Col$A$. Also, since

$$A\mathbf{w} = \begin{bmatrix} -8 & 5 & -2 & 0 \\ -5 & 2 & 1 & -2 \\ 10 & -8 & 6 & -3 \\ 3 & -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$, $\mathbf{w}$ is in Nul$A$.

51. **a**. To show that $\mathbf{a}_3$ and $\mathbf{a}_5$ are in the column space of B, we can row reduce the matrices $\begin{bmatrix} B & \mathbf{a}_3 \end{bmatrix}$ and $\begin{bmatrix} B & \mathbf{a}_5 \end{bmatrix}$: $\begin{bmatrix} B & \mathbf{a}_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} B & \mathbf{a}_5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 10/3 \\ 0 & 1 & 0 & -26/3 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Since both these systems are consistent, $\mathbf{a}_3$ and $\mathbf{a}_5$ are in the column space of B. Notice that the same conclusions can be drawn by observing the reduced row echelon form for A: $A \sim \begin{bmatrix} 1 & 0 & 1/3 & 0 & 10/3 \\ 0 & 1 & 1/3 & 0 & -26/3 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

b. We find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables by using the reduced row echelon form of A given above: $x_1 = (-1/3)x_3 - (10/3)x_5$, $x_2 = (-1/3)x_3 + (26/3)x_5$, $x_4 = 4x_5$

with x_3 and x_5 free. So $\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=x_3\begin{bmatrix}-1/3\\-1/3\\1\\0\\0\end{bmatrix}+x_5\begin{bmatrix}-10/3\\26/3\\0\\4\\1\end{bmatrix}$, and a spanning set for Nul A is

$$\left\{\begin{bmatrix}-1/3\\-1/3\\1\\0\\0\end{bmatrix},\begin{bmatrix}-10/3\\26/3\\0\\4\\1\end{bmatrix}\right\}.$$

c. The reduced row echelon form of A shows that the columns of A are linearly dependent and do not span $\mathbb{R}^4$. Thus by Theorem 12 in Section 1.9, T is neither one-to-one nor onto.

52. Since the line lies both in $H=\text{Span}\{\mathbf{v}_1,\mathbf{v}_2\}$ and in $K=\text{Span}\{\mathbf{v}_3,\mathbf{v}_4\}$, $\mathbf{w}$ can be written both as $c_1\mathbf{v}_1+c_2\mathbf{v}_2$ and $c_3\mathbf{v}_3+c_4\mathbf{v}_4$. To find $\boldsymbol{w}$ we must find the c_j's which solve $c_1\mathbf{v}_1+c_2\mathbf{v}_2-c_3\mathbf{v}_3-c_4\mathbf{v}_4=\mathbf{0}$. Row reduction of $[\mathbf{v}_1\quad\mathbf{v}_2\quad-\mathbf{v}_3\quad-\mathbf{v}_4\quad\mathbf{0}]$ yields

$\begin{bmatrix}5&1&-2&0&0\\3&3&1&12&0\\8&4&-5&28&0\end{bmatrix}\sim\begin{bmatrix}1&0&0&-10/3&0\\0&1&0&26/3&0\\0&0&1&-4&0\end{bmatrix}$, so the vector of c_j's must be a multiple of (10/3, –26/3, 4, 1). One simple choice is (10, –26, 12, 3), which gives $\mathbf{w}=10\mathbf{v}_1-26\mathbf{v}_2=12\mathbf{v}_3+3\mathbf{v}_4=(24,-48,-24)$. Another choice for $\mathbf{w}$ is (1, –2, –1).

4.3 - Linearly Independent Sets; Bases

Notes: The definition for basis is given initially for subspaces because this emphasizes that the basis elements must be in the subspace. Students often overlook this point when the definition is given for a vector space (see Exercise 35). The subsection on bases for Nul A and Col A is essential for Section 4.5. The subsection on “Two Views of a Basis” is also fundamental to understanding the interplay between linearly independent sets, spanning sets, and bases. Key exercises in this section are Exercises 41–45, which help to deepen students’ understanding of these different subsets of a vector space.

1. Consider the matrix whose columns are the given set of vectors. This 3×3 matrix is in echelon form, and has 3 pivot positions. Thus by the Invertible Matrix Theorem, its columns are linearly independent and span $\mathbb{R}^3$. So the given set of vectors is a basis for $\mathbb{R}^3$.

2. Since the zero vector is a member of the given set of vectors, the set cannot be linearly independent and thus cannot be a basis for $\mathbb{R}^3$. Now consider the matrix whose columns are the given set of vectors. This 3×3 matrix has only 2 pivot positions. Thus by the Invertible Matrix Theorem, its columns do not span $\mathbb{R}^3$.

3. Consider the matrix whose columns are the given set of vectors. The reduced echelon form of this matrix is $\begin{bmatrix} 1 & 3 & -3 \\ 0 & 2 & -5 \\ -2 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 9/2 \\ 0 & 1 & -5/2 \\ 0 & 0 & 0 \end{bmatrix}$ so the matrix has only two pivot positions. Thus its columns do not form a basis for $\mathbb{R}^3$; the set of vectors is neither linearly independent nor does it span $\mathbb{R}^3$.

4. Consider the matrix whose columns are the given set of vectors. The reduced echelon form of this matrix is $\begin{bmatrix} 2 & 1 & -7 \\ -2 & -3 & 5 \\ 1 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, so the matrix has three pivot positions. Thus its columns form a basis for $\mathbb{R}^3$.

5. Since the zero vector is a member of the given set of vectors, the set cannot be linearly independent and thus cannot be a basis for $\mathbb{R}^3$. Now consider the matrix whose columns are the given set of vectors. The reduced echelon form of this matrix is $\begin{bmatrix} 1 & -2 & 0 & 0 \\ -3 & 9 & 0 & -3 \\ 0 & 0 & 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, so the matrix has a pivot in each row. Thus the given set of vectors spans $\mathbb{R}^3$.

6. Consider the matrix whose columns are the given set of vectors. Since the matrix cannot have a pivot in each row, its columns cannot span $\mathbb{R}^3$; thus the given set of vectors is not a basis for $\mathbb{R}^3$. The reduced echelon form of the matrix is $\begin{bmatrix} 1 & -4 \\ 2 & -5 \\ -3 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, so the matrix has a pivot in each column. Thus the given set of vectors is linearly independent.

7. Consider the matrix whose columns are the given set of vectors. Since the matrix cannot have a pivot in each row, its columns cannot span $\mathbb{R}^3$; thus the given set of vectors is not a basis for $\mathbb{R}^3$. The reduced echelon form of the matrix is $\begin{bmatrix} -2 & 6 \\ 3 & -1 \\ 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, so the matrix has a pivot in each column. Thus the given set of vectors is linearly independent.

8. Consider the matrix whose columns are the given set of vectors. Since the matrix cannot have a pivot in each column, the set cannot be linearly independent and thus cannot be a basis for $\mathbb{R}^3$. The reduced echelon form of this matrix is $\begin{bmatrix} 1 & 0 & 3 & 0 \\ -4 & 3 & -5 & 2 \\ 3 & -1 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -3/2 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 1/2 \end{bmatrix}$, so the matrix has a pivot in each row. Thus the given set of vectors spans $\mathbb{R}^3$.

9. We find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables by using the reduced echelon form of A: $\begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -5 & 4 \\ 3 & -2 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -5 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. So $x_1 = 3x_3 - 2x_4$, $x_2 = 5x_3 - 4x_4$, with x_3 and x_4 free. So $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 5 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ -4 \\ 0 \\ 1 \end{bmatrix}$, and a basis for Nul A is $\left\{ \begin{bmatrix} 3 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\}$.

10. We find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables by using the reduced echelon form of A: $\begin{bmatrix} 1 & 0 & -5 & 1 & 4 \\ -2 & 1 & 6 & -2 & -2 \\ 0 & 2 & -8 & 1 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5 & 0 & 7 \\ 0 & 1 & -4 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}$. So $x_1 = 5x_3 - 7x_5$, $x_2 = 4x_3 - 6x_5$, $x_4 = 3x_5$, with x_3 and x_5 free. So $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -7 \\ -6 \\ 0 \\ 3 \\ 1 \end{bmatrix}$, and a basis for Nul A is $\left\{ \begin{bmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ -6 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}$.

11. Let $A = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$. Then we wish to find a basis for Nul A. We find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables: $x = -2y - z$ with y and z free. So $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and a basis for Nul A is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

12. We want to find a basis for the set of vectors in $\mathbb{R}^2$ in the line $5x - y = 0$. Let $A = \begin{bmatrix} 5 & -1 \end{bmatrix}$. Then we wish to find a basis for Nul A. We find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables: $y = 5x$ with x free. So $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 5 \end{bmatrix}$, and a basis for Nul A is $\left\{ \begin{bmatrix} 1 \\ 5 \end{bmatrix} \right\}$.

13. Since B is a row echelon form of A, we see that the first and second columns of A are its pivot columns. Thus a basis for Col A is $\left\{\begin{bmatrix}-2\\2\\-3\end{bmatrix},\begin{bmatrix}4\\-6\\8\end{bmatrix}\right\}$.

To find a basis for Nul A, we find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables: $x_1 = -6x_3 - 5x_4$, $x_2 = (-5/2)x_3 - (3/2)x_4$, with x_3 and x_4 free. So

$$\mathbf{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = x_3\begin{bmatrix}-6\\-5/2\\1\\0\end{bmatrix} + x_4\begin{bmatrix}-5\\-3/2\\0\\1\end{bmatrix}, \text{ and a basis for Nul } A \text{ is } \left\{\begin{bmatrix}-6\\-5/2\\1\\0\end{bmatrix},\begin{bmatrix}-5\\-3/2\\0\\1\end{bmatrix}\right\}.$$

A basis for Row A can be taken from the pivot rows of B: $\{[1\quad 0\quad 6\quad 5],[0\quad 2\quad 5\quad 3]\}$.

14. Since B is a row echelon form of A, we see that the first, third, and fifth columns of A are its pivot columns. Thus a basis for Col A is $\left\{\begin{bmatrix}1\\2\\1\\3\end{bmatrix},\begin{bmatrix}-5\\-5\\0\\-5\end{bmatrix},\begin{bmatrix}-3\\2\\5\\-2\end{bmatrix}\right\}$.

To find a basis for Nul A, we find the general solution of $A\mathbf{x} = \mathbf{0}$ in terms of the free variables, mentally completing the row reduction of B to get: $x_1 = -2x_2 - 4x_4$, $x_3 = (7/5)x_4$, $x_5 = 0$, with x_2 and x_4 free. So $\mathbf{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix} = x_2\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix} + x_4\begin{bmatrix}-4\\0\\7/5\\1\\0\end{bmatrix}$, and a basis for Nul A is $\left\{\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}-4\\0\\7/5\\1\\0\end{bmatrix}\right\}$.

A basis for Row A can be taken from the pivot rows of B:
$\{[1\quad 2\quad 0\quad 4\quad 5],[0\quad 0\quad 5\quad -7\quad 8],[0\quad 0\quad 0\quad 0\quad -9]\}$.

15. This problem is equivalent to finding a basis for Col A, where $A = [\mathbf{v}_1\quad \mathbf{v}_2\quad \mathbf{v}_3\quad \mathbf{v}_4\quad \mathbf{v}_5]$. Since the reduced echelon form of A is $\begin{bmatrix}1&0&-3&1&2\\0&1&-4&-3&1\\-3&2&1&-8&-6\\2&-3&6&7&9\end{bmatrix} \sim \begin{bmatrix}1&0&-3&0&4\\0&1&-4&0&-5\\0&0&0&1&-2\\0&0&0&0&0\end{bmatrix}$,

we see that the first, second, and fourth columns of A are its pivot columns. Thus a basis for the space spanned by the given vectors is $\left\{\begin{bmatrix}1\\0\\-3\\2\end{bmatrix},\begin{bmatrix}0\\1\\2\\-3\end{bmatrix},\begin{bmatrix}1\\-3\\-8\\7\end{bmatrix}\right\}$.

16. This problem is equivalent to finding a basis for Col A, where $A=[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4 \quad \mathbf{v}_5]$. Since the reduced echelon form of A is $\begin{bmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & -1 & 2 & 3 & -1 \\ 1 & 1 & -1 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 & -2 \\ 0 & 1 & 0 & -3 & 5 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, we see that the first, second, and third columns of A are its pivot columns. Thus a basis for the space spanned by the given vectors is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -1 \\ 2 \\ -1 \end{bmatrix} \right\}$.

17. This problem is equivalent to finding a basis for Col A, where $A=[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4 \quad \mathbf{v}_5]$. Since the reduced echelon form of A is $\begin{bmatrix} 8 & 4 & -1 & 6 & -1 \\ 9 & 5 & -4 & 8 & 4 \\ -3 & 1 & -9 & 4 & 11 \\ -6 & -4 & 6 & -7 & -8 \\ 0 & 4 & -7 & 10 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1/2 & 3 \\ 0 & 1 & 0 & 5/2 & -7 \\ 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, we see that the first, second, and third columns of A are its pivot columns. Thus a basis for the space spanned by the given vectors is $\left\{ \begin{bmatrix} 8 \\ 9 \\ -3 \\ -6 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 1 \\ -4 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ -9 \\ 6 \\ -7 \end{bmatrix} \right\}$.

18. This problem is equivalent to finding a basis for Col A, where $A=[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4 \quad \mathbf{v}_5]$. Since the reduced echelon form of A is $\begin{bmatrix} -8 & 8 & -8 & 1 & -9 \\ 7 & -7 & 7 & 4 & 3 \\ 6 & -9 & 4 & 9 & -4 \\ 5 & -5 & 5 & 6 & -1 \\ -7 & 7 & -7 & -7 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5/3 & 0 & 4/3 \\ 0 & 1 & 2/3 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, we see that the first, second, and fourth columns of A are its pivot columns. Thus a basis for the space spanned by the given vectors is $\left\{ \begin{bmatrix} -8 \\ 7 \\ 6 \\ 5 \\ -7 \end{bmatrix}, \begin{bmatrix} 8 \\ -7 \\ -9 \\ -5 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 9 \\ 6 \\ -7 \end{bmatrix} \right\}$.

19. Since $4\mathbf{v}_1 + 5\mathbf{v}_2 - 3\mathbf{v}_3 = \mathbf{0}$, we see that each of the vectors is a linear combination of the others. Thus the sets $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_3\}$, and $\{\mathbf{v}_2, \mathbf{v}_3\}$ all span H. Since we may confirm that none of the three vectors is a multiple of any of the others, the sets $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_3\}$, and $\{\mathbf{v}_2, \mathbf{v}_3\}$ are linearly independent and thus each forms a basis for *H*.

20. Since $\mathbf{v}_1 - 3\mathbf{v}_2 + 5\mathbf{v}_3 = \mathbf{0}$, we see that each of the vectors is a linear combination of the others. Thus the sets $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_3\}$, and $\{\mathbf{v}_2, \mathbf{v}_3\}$ all span *H*. Since we may confirm that none of the three vectors is a multiple of any of the others, the sets $\{\mathbf{v}_1, \mathbf{v}_2\}$, $\{\mathbf{v}_1, \mathbf{v}_3\}$, and $\{\mathbf{v}_2, \mathbf{v}_3\}$ are linearly independent and thus each forms a basis for *H*.

21. False. The zero vector by itself is linearly dependent. See the paragraph preceding Theorem 4.

22. False. The subspace spanned by the set must also coincide with *H*. See the definition of a basis.

23. False. The set $\{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ must also be linearly independent. See the definition of a basis.

24. True. Apply the Spanning Set Theorem to *V* instead of *H*. The space *V* is nonzero because the spanning set uses nonzero vectors.

25. True. See Example 3.

26. True. See the subsection "Two Views of a Basis."

27. False. See the subsection "Two Views of a Basis."

28. False. See the two paragraphs before Example 8

29. False. See the box before Example 9.

30. False. See the warning after Theorem 6.

31. False. See the footnote for Example 10.

32. True. See Theorem 7.

33. Let $A = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{v}_4]$. Then *A* is square and its columns span $\mathbb{R}^4$ since $\mathbb{R}^4 = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$. So its columns are linearly independent by the Invertible Matrix Theorem, and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a basis for $\mathbb{R}^4$.

34. Let $A = [\mathbf{v}_1 \quad \ldots \quad \mathbf{v}_n]$. Then *A* is square and its columns are linearly independent, so its columns span $\mathbb{R}^n$ by the Invertible Matrix Theorem. Thus $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$.

35. In order for the set to be a basis for *H*, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ must be a spanning set for *H*; that is, $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. The exercise shows that *H* is a subset of $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. but there are vectors in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ which are not in *H* ($\mathbf{v}_1$ and $\mathbf{v}_3$, for example). So $H \neq \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is not a basis for *H*.

36. Since $\sin t \cos t = (1/2) \sin 2t$, the set $\{\sin t, \sin 2t\}$ spans the subspace. By inspection we note that this set is linearly independent, so $\{\sin t, \sin 2t\}$ is a basis for the subspace.

37. The set {cos ωt, sin ωt} spans the subspace. By inspection we note that this set is linearly independent, so {cos ωt, sin ωt} is a basis for the subspace.

38. The set $\{e^{-bt}, te^{-bt}\}$ spans the subspace. By inspection we note that this set is linearly independent, so $\{e^{-bt}, te^{-bt}\}$ is a basis for the subspace.

39. Let A be the $n \times k$ matrix $[\mathbf{v}_1 \quad \ldots \quad \mathbf{v}_k]$. Since A has fewer columns than rows, there cannot be a pivot position in each row of A. By Theorem 4 in Section 1.4, the columns of A do not span $\mathbb{R}^n$ and thus are not a basis for $\mathbb{R}^n$.

40. Let A be the $n \times k$ matrix $[\mathbf{v}_1 \quad \ldots \quad \mathbf{v}_k]$. Since A has fewer rows than columns rows, there cannot be a pivot position in each column of A. By Theorem 8 in Section 1.6, the columns of A are not linearly independent and thus are not a basis for $\mathbb{R}^n$.

41. Suppose that $\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ is linearly dependent. Then there exist scalars $c_1,\ldots,c_p$ not all zero with $c_1\mathbf{v}_1+\ldots+c_p\mathbf{v}_p=\mathbf{0}$. Since T is linear, $T(c_1\mathbf{v}_1+\ldots+c_p\mathbf{v}_p)=c_1T(\mathbf{v}_1)+\ldots+c_pT(\mathbf{v}_p)$ and $T(c_1\mathbf{v}_1+\ldots+c_p\mathbf{v}_p)=T(\mathbf{0})=\mathbf{0}$.

Thus $c_1T(\mathbf{v}_1)+\ldots+c_pT(\mathbf{v}_p)=\mathbf{0}$ and since not all of the c_i are zero, $\{T(\mathbf{v}_1),\ldots,T(\mathbf{v}_p)\}$ is linearly dependent.

42. Suppose that $\{T(\mathbf{v}_1),\ldots,T(\mathbf{v}_p)\}$ is linearly dependent. Then there exist scalars $c_1,\ldots,c_p$ not all zero with $c_1T(\mathbf{v}_1)+\ldots+c_pT(\mathbf{v}_p)=\mathbf{0}$. Since T is linear, $T(c_1\mathbf{v}_1+\ldots+c_p\mathbf{v}_p)=c_1T(\mathbf{v}_1)+\ldots+c_pT(\mathbf{v}_p)=\mathbf{0}=T(\mathbf{0})$. Since T is one-to-one $T(c_1\mathbf{v}_1+\ldots+c_p\mathbf{v}_p)=T(\mathbf{0})$ implies that $c_1\mathbf{v}_1+\ldots+c_p\mathbf{v}_p=\mathbf{0}$. Since not all of the c_i are zero, $\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ is linearly dependent.

43. Neither polynomial is a multiple of the other polynomial. So $\{\mathbf{p}_1,\mathbf{p}_2\}$ is a linearly independent set in $\mathbb{P}_3$. Note: $\{\mathbf{p}_1,\mathbf{p}_2\}$ is also a linearly independent set in $\mathbb{P}_2$ since $\mathbf{p}_1$ and $\mathbf{p}_2$ both happen to be in $\mathbb{P}_2$.

44. By inspection, $\mathbf{p}_3=\mathbf{p}_1+\mathbf{p}_2$, or $\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3=\mathbf{0}$. By the Spanning Set Theorem, $\text{Span}\{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3\}=\text{Span}\{\mathbf{p}_1,\mathbf{p}_2\}$. Since neither $\mathbf{p}_1$ nor $\mathbf{p}_2$ is a multiple of the other, they are linearly independent and hence $\{\mathbf{p}_1,\mathbf{p}_2\}$ is a basis for $\text{Span}\{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3\}$.

45. Let $\{\mathbf{v}_1,\mathbf{v}_3\}$ be any linearly independent set in a vector space V, and let $\mathbf{v}_2$ and $\mathbf{v}_4$ each be linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_3$. For instance, let $\mathbf{v}_1=\mathbf{u}_1$ and $\mathbf{v}_2=\mathbf{u}_2$, then set $\mathbf{v}_2=5\mathbf{v}_1$ and $\mathbf{v}_4=\mathbf{v}_1+\mathbf{v}_3$. Then $\{\mathbf{v}_1,\mathbf{v}_3\}$ is a basis for $\text{Span}\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4\}$.

46. Row reduce the following matrices to identify their pivot columns:

$$[\mathbf{u}_1\ \ \mathbf{u}_2\ \ \mathbf{u}_3]=\begin{bmatrix}1&0&3\\2&2&4\\0&-1&1\\-1&1&-4\end{bmatrix}\sim\begin{bmatrix}1&0&3\\0&1&-1\\0&0&0\\0&0&0\end{bmatrix},\text{ so } \{\mathbf{u}_1,\mathbf{u}_2\} \text{ is a basis for } H.$$

$$[\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3]=\begin{bmatrix}-2&2&-1\\-2&3&4\\-1&2&6\\3&-6&-2\end{bmatrix}\sim\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix},\text{ so } \{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3\} \text{ is a basis for } K.$$

$$[\mathbf{u}_1\ \ \mathbf{u}_2\ \ \mathbf{u}_3\ \ \mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3]=\begin{bmatrix}1&0&3&-2&2&-1\\2&2&4&-2&3&4\\0&-1&1&-1&2&6\\-1&1&-4&3&-6&-2\end{bmatrix}\sim\begin{bmatrix}1&0&3&-2&0&0\\0&1&-1&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix},\text{ so}$$

$\{\mathbf{u}_1,\mathbf{u}_2,\mathbf{v}_2,\mathbf{v}_3\}$ is a basis for $H+K$.

47. For example, writing $c_1\cdot t+c_2\cdot \sin t+c_3\cos 2t+c_4\sin t\cos t=0$ with $t = 0, .1, .2, .3$ gives the following coefficent matrix A for the homogeneous system $A\mathbf{c}=\mathbf{0}$ (to four decimal places):

$$A=\begin{bmatrix}0&\sin 0&\cos 0&\sin 0\cos 0\\.1&\sin .1&\cos .2&\sin .1\cos .1\\.2&\sin .2&\cos .4&\sin .2\cos .2\\.3&\sin .3&\cos .6&\sin .3\cos .3\end{bmatrix}=\begin{bmatrix}0&0&1&0\\.1&.0998&.9801&.0993\\.2&.1987&.9211&.1947\\.3&.2955&.8253&.2823\end{bmatrix}.\text{ This matrix is invertible, so}$$

the system $A\mathbf{c}=\mathbf{0}$ has only the trivial solution and $\{t, \sin t, \cos 2t, \sin t\cos t\}$ is a linearly independent set of functions.

48. For example, writing $c_1\cdot 1+c_2\cdot\cos t+c_3\cdot\cos^2 t+c_4\cdot\cos^3 t+c_5\cdot\cos^4 t+c_6\cdot\cos^5 t+c_7\cdot\cos^6 t=0$ with $t = 0, .1, .2, .3, .4, .5, .6$ gives the following coefficent matrix A for the homogeneous system $A\mathbf{c}=\mathbf{0}$

$$\text{(to four decimal places): } A=\begin{bmatrix}1&\cos 0&\cos^2 0&\cos^3 0&\cos^4 0&\cos^5 0&\cos^6 0\\1&\cos .1&\cos^2 .1&\cos^3 .1&\cos^4 .1&\cos^5 .1&\cos^6 .1\\1&\cos .2&\cos^2 .2&\cos^3 .2&\cos^4 .2&\cos^5 .2&\cos^6 .2\\1&\cos .3&\cos^2 .3&\cos^3 .3&\cos^4 .3&\cos^5 .3&\cos^6 .3\\1&\cos .4&\cos^2 .4&\cos^3 .4&\cos^4 .4&\cos^5 .4&\cos^6 .4\\1&\cos .5&\cos^2 .5&\cos^3 .5&\cos^4 .5&\cos^5 .5&\cos^6 .5\\1&\cos .6&\cos^2 .6&\cos^3 .6&\cos^4 .6&\cos^5 .6&\cos^6 .6\end{bmatrix}$$

$$=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & .9950 & .9900 & .9851 & .9802 & .9753 & .9704 \\ 1 & .9801 & .9605 & .9414 & .9226 & .9042 & .8862 \\ 1 & .9553 & .9127 & .8719 & .8330 & .7958 & .7602 \\ 1 & .9211 & .8484 & .7814 & .7197 & .6629 & .6106 \\ 1 & .8776 & .7702 & .6759 & .5931 & .5205 & .4568 \\ 1 & .8253 & .6812 & .5622 & .4640 & .3830 & .3161 \end{bmatrix}$$. This matrix is invertible, so the system Ac = 0 has only the trivial solution and $\{1, \cos t, \cos^2 t, \cos^3 t, \cos^4 t, \cos^5 t, \cos^6 t\}$ is a linearly independent set of functions.

4.4 - Coordinate Systems

Notes: Section 4.6 depends heavily on this section, as does Section 5.4. It is possible to cover the $\mathbb{R}^n$ parts of the two later sections, however, if the first half of Section 4.4 (and perhaps Example 7) is covered. The linearity of the coordinate mapping is used in Section 5.4 to find the matrix of a transformation relative to a basis. The change-of-coordinates matrix appears in Section 5.4, Theorem 8 and Exercise 31. The concept of an isomorphism is needed in the proof of Theorem 17 in Section 4.8. Exercise 29 is used in Section 4.6 to show that the change-of-coordinates matrix is invertible.

1. We calculate that $\mathbf{x}=5\begin{bmatrix} 3 \\ -5 \end{bmatrix}+3\begin{bmatrix} -4 \\ 6 \end{bmatrix}=\begin{bmatrix} 3 \\ -7 \end{bmatrix}$.

2. We calculate that $\mathbf{x}=8\begin{bmatrix} 4 \\ 5 \end{bmatrix}+(-5)\begin{bmatrix} 6 \\ 7 \end{bmatrix}=\begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

3. We calculate that $\mathbf{x}=3\begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}+0\begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix}+(-1)\begin{bmatrix} 4 \\ -7 \\ 0 \end{bmatrix}=\begin{bmatrix} -1 \\ -5 \\ 9 \end{bmatrix}$.

4. We calculate that $\mathbf{x}=(-4)\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}+8\begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix}+(-7)\begin{bmatrix} 4 \\ -7 \\ 3 \end{bmatrix}=\begin{bmatrix} 0 \\ 1 \\ -5 \end{bmatrix}$.

5. The matrix $[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{x}]$ row reduces to $\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & -5 \end{bmatrix}$, so $[\mathbf{x}]_B=\begin{bmatrix} 8 \\ -5 \end{bmatrix}$.

6. The matrix $[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{x}]$ row reduces to $\begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 2 \end{bmatrix}$, so $[\mathbf{x}]_B=\begin{bmatrix} -6 \\ 2 \end{bmatrix}$.

7. The matrix $[\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3 \quad \mathbf{x}]$ row reduces to $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$, so $[\mathbf{x}]_B=\begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$.

8. The matrix $\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{x}\end{bmatrix}$ row reduces to $\begin{bmatrix}1 & 0 & 0 & -2\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 5\end{bmatrix}$, so $[\mathbf{x}]_B = \begin{bmatrix}-2\\ 0\\ 5\end{bmatrix}$.

9. The change-of-coordinates matrix from B to the standard basis in $\mathbb{R}^2$ is $P_B = \begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2\end{bmatrix} = \begin{bmatrix}2 & 1\\ -9 & 8\end{bmatrix}$.

10. The change-of-coordinates matrix from B to the standard basis in $\mathbb{R}^3$ is

$$P_B = \begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3\end{bmatrix} = \begin{bmatrix}3 & 2 & 8\\ -1 & 0 & -2\\ 4 & -5 & 7\end{bmatrix}.$$

11. Since P_B^{-1} converts $\mathbf{x}$ into its B-coordinate vector, we find that

$$[\mathbf{x}]_B = P_B^{-1}\mathbf{x} = \begin{bmatrix}3 & -4\\ -5 & 6\end{bmatrix}^{-1}\begin{bmatrix}2\\ -6\end{bmatrix} = \begin{bmatrix}-3 & -2\\ -5/2 & -3/2\end{bmatrix}\begin{bmatrix}2\\ -6\end{bmatrix} = \begin{bmatrix}6\\ 4\end{bmatrix}.$$

12. Since P_B^{-1} converts $\mathbf{x}$ into its B-coordinate vector, we find that

$$[\mathbf{x}]_B = P_B^{-1}\mathbf{x} = \begin{bmatrix}4 & 6\\ 5 & 7\end{bmatrix}^{-1}\begin{bmatrix}2\\ 0\end{bmatrix} = \begin{bmatrix}-7/2 & 3\\ 5/2 & -2\end{bmatrix}\begin{bmatrix}2\\ 0\end{bmatrix} = \begin{bmatrix}-7\\ 5\end{bmatrix}.$$

13. We must find c_1, c_2, and c_3 such that $c_1(1+t^2)+c_2(t+t^2)+c_3(1+2t+t^2) = \mathbf{p}(t) = 1+4t+7t^2$.
Equating the coefficients of the two polynomials produces the system of equations

$$\begin{aligned} c_1 \quad + \quad\quad c_3 &= 1\\ c_2 + 2c_3 &= 4\\ c_1 + c_2 + c_3 &= 7\end{aligned}$$

. We row reduce the augmented matrix for the system of equations to find

$$\begin{bmatrix}1 & 0 & 1 & 1\\ 0 & 1 & 2 & 4\\ 1 & 1 & 1 & 7\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 0 & 2\\ 0 & 1 & 0 & 6\\ 0 & 0 & 1 & -1\end{bmatrix}, \text{ so } [\mathbf{p}]_B = \begin{bmatrix}2\\ 6\\ -1\end{bmatrix}.$$

One may also solve this problem using the coordinate vectors of the given polynomials relative to the standard basis $\{1, t, t^2\}$; the same system of linear equations results.

14. We must find c_1, c_2, and c_3 such that $c_1(1-t^2)+c_2(t-t^2)+c_3(2-2t+t^2) = \mathbf{p}(t) = 3+t-6t^2$.
Equating the coefficients of the two polynomials produces the system of equations

$$\begin{aligned} c_1 \quad + \quad\quad 2c_3 &= 3\\ c_2 - 2c_3 &= 1\\ -c_1 - c_2 + c_3 &= -6\end{aligned}$$

. We row reduce the augmented matrix for the system of equations to

$$\text{find } \begin{bmatrix}1 & 0 & 2 & 3\\ 0 & 1 & -2 & 1\\ -1 & -1 & 1 & -6\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 0 & 7\\ 0 & 1 & 0 & -3\\ 0 & 0 & 1 & -2\end{bmatrix}, \text{ so } [\mathbf{p}]_B = \begin{bmatrix}7\\ -3\\ -2\end{bmatrix}.$$

One may also solve this problem using the coordinate vectors of the given polynomials relative to the standard basis $\{1, t, t^2\}$; the same system of linear equations results.

15. True. See the definition of the B-coordinate vector.

16. True. See Example 2.

17. False. See Equation (4).

18. False. By definition, the coordinate mapping goes in the opposite direction.

19. False. $\mathbb{P}_3$ is isomorphic to $\mathbb{R}^4$. See Example 5.

20. True. If the plane passes through the origin, as in Example 7, the plane is isomorphic to $\mathbb{R}^2$.

21. We must solve the vector equation $x_1\begin{bmatrix}1\\-3\end{bmatrix}+x_2\begin{bmatrix}2\\-8\end{bmatrix}+x_3\begin{bmatrix}-3\\7\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$. We row reduce the augmented matrix for the system of equations to find $\begin{bmatrix}1&2&-3&1\\-3&-8&7&1\end{bmatrix}\sim\begin{bmatrix}1&0&-5&5\\0&1&1&-2\end{bmatrix}$. Thus we can let $x_1=5+5x_3$ and $x_2=-2-x_3$, where x_3 can be any real number. Letting $x_3=0$ and $x_3=1$ produces two different ways to express $\begin{bmatrix}1\\1\end{bmatrix}$ as a linear combination of the other vectors: $5\mathbf{v}_1-2\mathbf{v}_2$ and $10\mathbf{v}_1-3\mathbf{v}_2+\mathbf{v}_3$. There are infintely many correct answers to this problem.

22. For each k, $\mathbf{b}_k=0\cdot\mathbf{b}_1+\cdots+1\cdot\mathbf{b}_k+\cdots+0\cdot\mathbf{b}_n$, so $[\mathbf{b}_k]_B=\begin{bmatrix}0\\\vdots\\0\\1\\0\\\vdots\\0\end{bmatrix}=\mathbf{e}_k$.

23. The set S spans V because every $\mathbf{x}$ in V has a representation as a (unique) linear combination of elements in S. To show linear independence, suppose that $S=\{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$ and that $c_1\mathbf{v}_1+\cdots+c_n\mathbf{v}_n=\mathbf{0}$ for some scalars c_1, …, c_n. The case when $c_1=\cdots=c_n=0$ is one possibility. By hypothesis, this is the unique (and thus the only) possible representation of the zero vector as a linear combination of the elements in S. So S is linearly independent and is thus a basis for V.

24. For $\boldsymbol{w}$ in V there exist scalars k_1, k_2, k_3, and k_4 such that $\mathbf{w}=k_1\mathbf{v}_1+k_2\mathbf{v}_2+k_3\mathbf{v}_3+k_4\mathbf{v}_4$ because $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4\}$ spans V. Because the set is linearly dependent, there exist scalars c_1, c_2, c_3, and c_4 not all zero, such that $\mathbf{0}=c_1\mathbf{v}_1+c_2\mathbf{v}_2+c_3\mathbf{v}_3+c_4\mathbf{v}_4$. Adding there two equations gives $\mathbf{w}=\mathbf{w}+\mathbf{0}=(k_1+c_1)\mathbf{v}_1+(k_2+c_2)\mathbf{v}_2+(k_3+c_3)\mathbf{v}_3+(k_4+c_4)\mathbf{v}_4$. At least one of the weights in the third equation differs from the corresponding weight in the first equation because at least one of the

c_i is nonzero. So **w** is expressed in more than one way as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$.

25. The matrix of the transformation will be $P_B^{-1} = \begin{bmatrix} 1 & -2 \\ -4 & 9 \end{bmatrix}^{-1} = \begin{bmatrix} 9 & 2 \\ 4 & 1 \end{bmatrix}$.

26. The matrix of the transformation will be $P_B^{-1} = [\mathbf{b}_1 \quad \cdots \quad \mathbf{b}_n]^{-1}$.

27. Suppose that $[\mathbf{u}]_B = [\mathbf{w}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$. By definition of coordinate vectors, $\mathbf{u} = \mathbf{w} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$.

Since **u** and **w** were arbitrary elements of V, the coordinate mapping is one-to-one.

28. Given $\mathbf{y} = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$, let $\mathbf{u} = y_1\mathbf{b}_1 + \cdots + y_n\mathbf{b}_n$. Then, by definition, $[\mathbf{u}]_B = \mathbf{y}$. Since **y** was arbitrary, the coordinate mapping is onto $\mathbb{R}^n$.

29. Since the coordinate mapping is one-to-one, the following equations have the same solutions $c_1, \ldots, c_p$:

$c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p = \mathbf{0}$ (the zero vector in V) and $\left[c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p\right]_B = [\mathbf{0}]_B$ (the zero vector in $\mathbb{R}^n$). Since the coordina

linearly independent if and only if $\{[\mathbf{u}_1]_B, \ldots, [\mathbf{u}_p]_B\}$ is linearly independent. This result also follows directly from Exercises 43 and 44 in Section 4.3.

30. By definition, **w** is a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_p$ if and only if there exist scalars $c_1, \ldots, c_p$ such that

$\mathbf{w} = c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p$. Since the coordinate mapping is linear, $[\mathbf{w}]_B = c_1[\mathbf{u}_1]_B + \cdots + c_p[\mathbf{u}_p]_B$.
Conversely, the second equation implies the first equation because the coordinate mapping is one-to-one. Thus **w** is a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_p$ if and only if $[\mathbf{w}]_B$ is a linear combination of $[\mathbf{u}]_1, \ldots, [\mathbf{u}]_p$.

Note: Students need to be urged to *write* not just to compute in Exercises 31–37. The language in the *Study Guide* solution of Exercise 35 provides a model for the students. In Exercise 36, students may have difficulty distinguishing between the two isomorphic vector spaces, sometimes giving a vector in $\mathbb{R}^3$ as an answer for part (b).

31. The coordinate mapping produces the coordinate vectors (1, 0, 0, 2), (2, 1, –3, 0), and (0, –1, 2, –1) respectively. We test for linear independence of these vectors by writing them as columns of a matrix and row reducing: $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & -3 & 2 \\ 2 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Since the matrix has a pivot in each column, its columns (and thus the given polynomials) are linearly independent.

32. The coordinate mapping produces the coordinate vectors (1, 0, –2, –1), (0, 1, 0, 2), and (1, 1, –2, 0) respectively. We test for linear independence of these vectors by writing them as columns of a matrix and row reducing: $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -2 & 0 & -2 \\ -1 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Since the matrix has a pivot in each column, its columns (and thus the given polynomials) are linearly independent.

33. The coordinate mapping produces the coordinate vectors (1, –2, 1, 0), (0, 1, –2, 1), and (1, –3, 3, –1) respectively. We test for linear independence of these vectors by writing them as columns of a matrix and row reducing: $\begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & -3 \\ 1 & -2 & 3 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Since the matrix does not have a pivot in each column, its columns (and thus the given polynomials) are linearly dependent.

34. The coordinate mapping produces the coordinate vectors (8, –12, 6, –1), (9, –6, 1, 0), and (1, 6, –5,1) respectively. We test for linear independence of these vectors by writing them as columns of a matrix and row reducing: $\begin{bmatrix} 8 & 9 & 1 \\ -12 & -6 & 6 \\ 6 & 1 & -5 \\ -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Since the matrix does not have a pivot in each column, its columns (and thus the given polynomials) are linearly dependent.

35. In each part, place the coordinate vectors of the polynomials into the columns of a matrix and reduce the matrix to echelon form.

a. $\begin{bmatrix} 1 & -3 & -4 & 1 \\ -3 & 5 & 5 & 0 \\ 5 & -7 & -6 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -4 & 1 \\ 0 & -4 & -7 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Since there is not a pivot in each row, the original four column vectors do not span $\mathbb{R}^3$. By the isomorphism between $\mathbb{R}^3$ and $\mathbb{P}_2$, the given set of polynomials does not span $\mathbb{P}_2$.

b. $\begin{bmatrix} 0 & 1 & -3 & 2 \\ 5 & -8 & 4 & -3 \\ 1 & -2 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & 0 \\ 0 & 2 & -6 & -3 \\ 0 & 0 & 0 & 7/2 \end{bmatrix}$. Since there is a pivot in each row, the original four column vectors span $\mathbb{R}^3$. By the isomorphism between $\mathbb{R}^3$ and $\mathbb{P}_2$, the given set of polynomials spans $\mathbb{P}_2$.

36. **a**. Place the coordinate vectors of the polynomials into the columns of a matrix and reduce the matrix to echelon form: $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$. The resulting matrix is invertible since it row equivalent to I_3. The original three column vectors form a basis for $\mathbb{R}^3$ by the Invertible Matrix Theorem. By the isomorphism between $\mathbb{R}^3$ and $\mathbb{P}_2$, the corresponding polynomials form a basis for $\mathbb{P}_2$.

b. Since $[\mathbf{q}]_B = (-1, 1, 2)$, $\mathbf{q} = -\mathbf{p}_1 + \mathbf{p}_2 + 2\mathbf{p}_3$. One might do the algebra in $\mathbb{P}_2$ or choose to compute $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & -3 & -3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -10 \end{bmatrix}$. This combination of the columns of the matrix corresponds to the same combination of $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$. So $\mathbf{q}(t) = 1 + 3t - 10t^2$.

37. The coordinate mapping produces the coordinate vectors (3, 7, 0, 0), (5, 1, 0, –2), (0, 1, –2, 0) and (1, 16, –6, 2) respectively. To determine whether the set of polynomials is a basis for $\mathbb{P}_3$, we investigate whether the coordinate vectors form a basis for $\mathbb{R}^4$. Writing the vectors as the columns of a matrix and row reducing $\begin{bmatrix} 3 & 5 & 0 & 1 \\ 7 & 1 & 1 & 16 \\ 0 & 0 & -2 & -6 \\ 0 & -2 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, we find that the matrix is not row equivalent to I_4. Thus the coordinate vectors do not form a basis for $\mathbb{R}^4$. By the isomorphism between $\mathbb{R}^4$ and $\mathbb{P}_3$, the given set of polynomials does not form a basis for $\mathbb{P}_3$.

38. The coordinate mapping produces the coordinate vectors (5, –3, 4, 2), (9, 1, 8, –6), (6, –2, 5, 0), and (0, 0, 0, 1) respectively. To determine whether the set of polynomials is a basis for $\mathbb{P}_3$, we investigate whether the coordinate vectors form a basis for $\mathbb{R}^4$. Writing the vectors as the columns of a matrix, and row reducing $\begin{bmatrix} 5 & 9 & 6 & 0 \\ -3 & 1 & -2 & 0 \\ 4 & 8 & 5 & 0 \\ 2 & -6 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3/4 & 0 \\ 0 & 1 & 1/4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ we find that the matrix is not row equivalent to I_4. Thus the coordinate vectors do not form a basis for $\mathbb{R}^4$. By the isomorphism between $\mathbb{R}^4$ and $\mathbb{P}_3$, the given set of polynomials does not form a basis for $\mathbb{P}_3$.

39. To show that $\mathbf{x}$ is in $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, we must show that the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{x}$ has a solution. The augmented matrix $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{x}]$ may be row reduced to show

$\begin{bmatrix} 11 & 14 & 19 \\ -5 & -8 & -13 \\ 10 & 13 & 18 \\ 7 & 10 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -5/3 \\ 0 & 1 & 8/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Since this system has a solution, x is in H. The solution allows us to find the B-coordinate vector for x: since $\mathbf{x} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = (-5/3)\mathbf{v}_1 + (8/3)\mathbf{v}_2$, $[\mathbf{x}]_B = \begin{bmatrix} -5/3 \\ 8/3 \end{bmatrix}$.

40. To show that $\mathbf{x}$ is in $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, we must show that the vector equation $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = \mathbf{x}$ has a solution. The augmented matrix $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{x}]$ may be row reduced to show

$\begin{bmatrix} -6 & 8 & -9 & 4 \\ 4 & -3 & 5 & 7 \\ -9 & 7 & -8 & -8 \\ 4 & -3 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The first three columns show that B is a basis for H.

Moreover, since this system has a solution, $\mathbf{x}$ is in H. The solution allows us to find the B-coordinate vector for $\mathbf{x}$: since $\mathbf{x} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + x_3\mathbf{v}_3 = 3\mathbf{v}_1 + 5\mathbf{v}_2 + 2\mathbf{v}_3$, $[\mathbf{x}]_B = \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}$.

41. We are given that $[\mathbf{x}]_B = \begin{bmatrix} 1/2 \\ 1/4 \\ 1/6 \end{bmatrix}$, where $B = \left\{ \begin{bmatrix} 2.6 \\ -1.5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4.8 \end{bmatrix} \right\}$. To find the coordinates of x relative to the standard basis in $\mathbb{R}^3$, we must find $\mathbf{x}$. We compute that

$$\mathbf{x} = P_B[\mathbf{x}]_B = \begin{bmatrix} 2.6 & 0 & 0 \\ -1.5 & 3 & 0 \\ 0 & 0 & 4.8 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/4 \\ 1/6 \end{bmatrix} = \begin{bmatrix} 1.3 \\ 0 \\ 0.8 \end{bmatrix}.$$

42. We are given that $[\mathbf{x}]_B = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/3 \end{bmatrix}$, where $B = \left\{ \begin{bmatrix} 2.6 \\ -1.5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4.8 \end{bmatrix} \right\}$. To find the coordinates of $\mathbf{x}$ relative to the standard basis in $\mathbb{R}^3$, we must find $\mathbf{x}$. We compute that

$$\mathbf{x} = P_B[\mathbf{x}]_B = \begin{bmatrix} 2.6 & 0 & 0 \\ -1.5 & 3 & 0 \\ 0 & 0 & 4.8 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 1.3 \\ 0.75 \\ 1.6 \end{bmatrix}.$$

4.5 - The Dimension of a Vector Space

Notes: Theorem 10 is true because a vector space isomorphic to $\mathbb{R}^n$ has the same algebraic properties as $\mathbb{R}^n$; a proof of this result may not be needed to convince the class. The proof of Theorem 10 relies upon the fact that the coordinate mapping is a linear transformation (which is Theorem 9 in Section 4.4). If you have skipped this result, you can prove Theorem 10 as is done in *Introduction to Linear Algebra* by Serge Lang (Springer-Verlag, New York, 1986). There are two separate groups of true-false questions in this section; the second batch is more theoretical in nature. Example 4 is useful to get students to visualize subspaces of different dimensions, and to see the relationships between subspaces of different dimensions. Exercises 51 and 52 investigate the relationship between the dimensions of the domain and the range of a linear transformation; Exercise 52 is mentioned in the proof of Theorem 20 in Section 4.8.

1. This subspace is $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are not multiples of each other, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent and is thus a basis for *H*. Hence the dimension of *H* is 2.

2. This subspace is $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1 = \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are not multiples of each other, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent and is thus a basis for *H*. Hence the dimension of *H* is 2.

3. This subspace is $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \end{bmatrix}$. Theorem 4 in Section 4.3 can be used to show that this set is linearly independent: $\mathbf{v}_1 \neq \mathbf{0}$, $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$, and (since its first entry is not zero) $\mathbf{v}_3$ is not a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent and is thus a basis for *H*. Alternatively, one can show that this set is linearly independent by row reducing the matrix $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3 \quad \mathbf{0}]$. Hence the dimension of the subspace is 3.

4. This subspace is $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ -1 \end{bmatrix}$. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are not multiples of each other, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent and is thus a basis for *H*. Hence the dimension of *H* is 2.

5. This subspace is $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 5 \\ 0 \\ 7 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} -2 \\ -4 \\ 2 \\ 6 \end{bmatrix}$. Since $\mathbf{v}_3 = -2\mathbf{v}_1$, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent. By the Spanning Set Theorem, $\mathbf{v}_3$ may be removed from the set with no change in the span of the set, so $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are not multiples of each other, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent and is thus a basis for *H*. Hence the dimension of *H* is 2.

6. This subspace is $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ -9 \\ -3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 6 \\ -2 \\ 5 \\ 1 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} -1 \\ -2 \\ 3 \\ 1 \end{bmatrix}$. Since $\mathbf{v}_3 = -(1/3)\mathbf{v}_1$, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent. By the Spanning Set Theorem, $\mathbf{v}_3$ may be removed from the set with no change in the span of the set, so $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are not multiples of each other, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent and is thus a basis for *H*. Hence the dimension of *H* is 2.

7. This subspace is H = Nul A, where $A = \begin{bmatrix} 1 & -3 & 1 \\ 0 & 1 & -2 \\ 0 & 2 & -1 \end{bmatrix}$. Since $\begin{bmatrix} A & \mathbf{0} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, the homogeneous system has only the trivial solution. Thus H = Nul $A = \{\mathbf{0}\}$, and the dimension of H is 0.

8. From the equation $a - 3b + c = 0$, it is seen that $(a, b, c, d) = b(3, 1, 0, 0) + c(-1, 0, 1, 0) + d(0, 0, 0, 1)$. Thus the subspace is $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where $\mathbf{v}_1 = (3,1,0,0)$, $\mathbf{v}_2 = (-1,0,1,0)$, and $\mathbf{v}_3 = (0,0,0,1)$. It is easily checked that this set of vectors is linearly independent, either by appealing to Theorem 4 in Section 4.3, or by row reducing $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{0} \end{bmatrix}$. Hence the dimension of the subspace is 3.

9. The matrix *A* with these vectors as its columns row reduces to

$$\begin{bmatrix} 1 & 3 & 9 & -7 \\ 0 & 1 & 4 & -3 \\ 2 & 1 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

There are two pivot columns, so the dimension of Col *A* (which is the dimension of the subspace spanned by the vectors) is 2.

10. The matrix *A* with these vectors as its columns row reduces to

$$\begin{bmatrix} 1 & -3 & -8 & -3 \\ -2 & 4 & 6 & 0 \\ 0 & 1 & 5 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

There are three pivot columns, so the dimension of Col *A* (which is the dimension of the subspace spanned by the vectors) and Row *A* is 3.

11. The matrix A is in echelon form. There are three pivot columns, so the dimension of Col A and Row A is 3. There are two columns without pivots, so the equation $A\mathbf{x} = \mathbf{0}$ has two free variables. Thus the dimension of Nul A is 2.

12. The matrix A is in echelon form. There are three pivot columns, so the dimension of Col A and Row A is 3. There are three columns without pivots, so the equation $A\mathbf{x} = \mathbf{0}$ has three free variables. Thus the dimension of Nul A is 3.

13. The matrix A is in echelon form. There are two pivot columns, so the dimension of Col A is 2. There are two columns without pivots, so the equation $A\mathbf{x} = \mathbf{0}$ has two free variables. Thus the dimension of Nul A is 2.

14. The matrix A row reduces to $\begin{bmatrix} 3 & 4 \\ -6 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. There are two pivot columns, so the dimension of Col A and Row A is 2. There are no columns without pivots, so the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{0}$. Thus Nul $A = \{\mathbf{0}\}$, and the dimension of Nul A is 0.

15. The matrix A is in echelon form. There are three pivot columns, so the dimension of Col A and Row A is 3. There are no columns without pivots, so the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{0}$. Thus Nul $A = \{\mathbf{0}\}$, and the dimension of Nul A is 0.

16. The matrix A is in echelon form. There are two pivot columns, so the dimension of Col A and Row A is 2. There is one column without a pivot, so the equation $A\mathbf{x} = \mathbf{0}$ has one free variable. Thus the dimension of Nul A is 1.

17. True. See the box before Theorem 14.

18. False. The number of **free** variables is equal to the dimension of Nul A; see the box before Theorem 14.

19. False. The plane must pass through the origin; see Example 4.

20. False. The dimension of $\mathbb{P}_n$ is $n + 1$; see Example 1.

21. False. Create a linearly independent set of signals with more than 10 vectors.

22. True. See the Rank Theorem.

23. False. Review the warning after Theorem 6 in Section 4.3.

24. True. See the remark in the proof of the Rank Theorem.

25. True. See Practice Problem 1.

26. False. A basis could still have only finitely many elements, which would make the vector space finite-dimensional.

27. The matrix whose columns are the coordinate vectors of the Hermite polynomials relative to the standard basis $\{1, t, t^2, t^3\}$ of $\mathbb{P}_3$ is $A = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -12 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix}$. This matrix has 4 pivots, so its columns are linearly independent. Since their coordinate vectors form a linearly independent set, the Hermite polynomials themselves are linearly independent in $\mathbb{P}_3$. Since there are four Hermite polynomials and dim $\mathbb{P}_3$ = 4, the Basis Theorem states that the Hermite polynomials form a basis for $\mathbb{P}_3$.

28. The matrix whose columns are the coordinate vectors of the Laguerre polynomials relative to the standard basis $\{1, t, t^2, t^3\}$ of $\mathbb{P}_3$ is $A = \begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & -1 & -4 & -18 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & -1 \end{bmatrix}$. This matrix has 4 pivots, so its columns are linearly independent. Since their coordinate vectors form a linearly independent set, the Laguerre polynomials themselves are linearly independent in $\mathbb{P}_3$. Since there are four Laguerre polynomials and dim $\mathbb{P}_3$ = 4, the Basis Theorem states that the Laguerre polynomials form a basis for $\mathbb{P}_3$.

29. The coordinates of $\mathbf{p}(t) = 7 - 12t - 8t^2 + 12t^3$ with respect to *B* satisfy
$c_1(1) + c_2(2t) + c_3(-2 + 4t^2) + c_4(-12t + 8t^3) = 7 - 12t - 8t^2 + 12t^3$.

Equating coefficients of like powers of *t* produces the system of equations

$$\begin{array}{rcrcrcrcr} c_1 & & & - & 2c_3 & & & = & 7 \\ & & 2c_2 & & & - & 12c_4 & = & -12 \\ & & & & 4c_3 & & & = & -8 \\ & & & & & & 8c_4 & = & 12 \end{array}.$$

Solving this system gives $c_1 = 3,\ c_2 = 3,\ c_3 = -2,\ c_4 = 3/2,$ and $[\mathbf{p}]_B = \begin{bmatrix} 3 \\ 3 \\ -2 \\ 3/2 \end{bmatrix}$.

30. The coordinates of $\mathbf{p}(t) = 7 - 8t + 3t^2$ with respect to *B* satisfy

$c_1(1) + c_2(1-t) + c_3(2 - 4t + t^2) = 7 - 8t + 3t^2$

Equating coefficients of like powers of *t* produces the system of equations

$$\begin{array}{rcrcrcr} c_1 & + & c_2 & + & 2c_3 & = & 7 \\ & & -c_2 & - & 4c_3 & = & -8 \\ & & & & c_3 & = & 3 \end{array}$$

Solving this system gives $c_1 = 5,\ c_2 = -4,\ c_3 = 3,$ and $[\mathbf{p}]_B = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$.

31. Note first that $n \geq 1$ since S cannot have fewer than 1 vector. Since $n \geq 1$, $V \neq \mathbf{0}$. Suppose that S spans V and that S contains fewer than n vectors. By the Spanning Set Theorem, some subset S' of S is a basis for V. Since S contains fewer than n vectors, and S' is a subset of S, S' also contains fewer than n vectors. Thus there is a basis S' for V with fewer than n vectors, but this is impossible by Theorem 10 since $\dim V = n$. Thus S cannot span V.

32. If $\dim V = \dim H = 0$, then $V = \{\mathbf{0}\}$ and $H = \{\mathbf{0}\}$, so $H = V$. Suppose that $\dim V = \dim H > 0$. Then H contains a basis S consisting of n vectors. But applying the Basis Theorem to V, S is also a basis for V. Thus $H = V = \text{Span } S$.

33. $\text{Nullity } A = 8 - 3 = 5,\ \text{rank } A = 3 = \text{rank } A^T$.

34. $\text{Nullity } A = 3 - 3 = 0,\ \text{rank } A = 3 = \text{rank } A^T$.

35. Since A has 4 rows and four pivot columns, $\text{Col } A = \mathbb{R}^4$, however since A has 7 colunmns Nul A is a 3 dimenional subspace of $\mathbb{R}^7$.

36. Since A has 5 rows and four pivot columns, $\text{Col } A$ is a 4 dimensional subspace of $\mathbb{R}^5$ with nullity $A = 6 - 4 = 2$.

37. $\text{Rank } A = 6 - 4 = 2$ so the dimension of the column space and row space is 2.

38. $\text{Rank } A = 6 - 5 = 1$ so the dimension of the column space and row space is 1.

39. In both cases there can be at most 5 pivots and so the maximum rank is 5.

40. In both cases there can be at most 3 pivots and so the row space can have dimension at most 3.

41. Suppose that $\dim \mathbb{P} = k < \infty$. Now $\mathbb{P}_n$ is a subspace of $\mathbb{P}$ for all n, and $\dim \mathbb{P}_{k-1} = k$, so $\dim \mathbb{P}_{k-1} = \dim \mathbb{P}$. This would imply that $\mathbb{P}_{k-1} = \mathbb{P}$, which is clearly untrue: for example $\mathbf{p}(t) = t^k$ is in $\mathbb{P}$ but not in $\mathbb{P}_{k-1}$. Thus the dimension of $\mathbb{P}$ cannot be finite.

42. The space $C(\mathbb{R})$ contains $\mathbb{P}$ as a subspace. If $C(\mathbb{R})$ were finite-dimensional, then $\mathbb{P}$ would also be finite-dimensional by Theorem 12. But $\mathbb{P}$ is infinite-dimensional by Exercise 41, so $C(\mathbb{R})$ must also be infinite-dimensional.

43. True. Apply the Spanning Set Theorem to the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ and produce a basis for V. This basis will not have more than p elements in it, so $\dim V \leq p$.

44. False. For a counterexample, let $\mathbf{v}$ be a non-zero vector in $\mathbb{R}^3$, and consider the set $\{\mathbf{v}, 2\mathbf{v}\}$. This is a linearly dependent set in $\mathbb{R}^3$, but $\dim \mathbb{R}^3 = 3 > 2$.

45. True. By Theorem 12, $\{\mathbf{v}_1,\ldots,\mathbf{v}_p\}$ can be expanded to find a basis for V. This basis will have at least p elements in it, so $\dim V \geq p$.

46. True. Take any basis (which will contain p vectors) for V and adjoin the zero vector to it.

47. True. If $\dim V \leq p$, there is a basis for V with p or fewer vectors. This basis would be a spanning set for V with p or fewer vectors, which contradicts the assumption.

48. False. For a counterexample, let $\mathbf{v}$ be a non-zero vector in $\mathbb{R}^3$, and consider the set $\{\mathbf{v}, 2\mathbf{v}\}$. This is a linearly dependent set in $\mathbb{R}^3$ with $3 - 1 = 2$ vectors, and $\dim \mathbb{R}^3 = 3$.

49. Since A is an $m \times n$ matrix and dim Row A = rank A,

dim Row A + nullity A = rank A + nullity A = n.

50. Since A^T is an $n \times m$ matrix and $\dim \text{Row } A = \dim \text{Col } A^T = \text{rank } A^T$,

$\dim \text{Col } A + \text{nullity } A^T = \text{rank } A^T + \text{nullity } A^T = m.$

51. Since H is a nonzero subspace of a finite-dimensional vector space V, H is finite-dimensional and has a basis. Let $\{\mathbf{u}_1,\ldots,\mathbf{u}_p\}$ be a basis for H. We show that the set $\{T(\mathbf{u}_1),\ldots,T(\mathbf{u}_p)\}$ spans $T(H)$. Let $\mathbf{y}$ be in $T(H)$. Then there is a vector $\mathbf{x}$ in H with $T(\mathbf{x}) = \mathbf{y}$. Since $\mathbf{x}$ is in H and $\{\mathbf{u}_1,\ldots,\mathbf{u}_p\}$ is a basis for H, $\mathbf{x}$ may be written as $\mathbf{x} = c_1\mathbf{u}_1 + \ldots + c_p\mathbf{u}_p$ for some scalars $c_1,\ldots,c_p$. Since the transformation T is linear, $\mathbf{y} = T(\mathbf{x}) = T(c_1\mathbf{u}_1 + \ldots + c_p\mathbf{u}_p) = c_1T(\mathbf{u}_1) + \ldots + c_pT(\mathbf{u}_p)$. Thus $\mathbf{y}$ is a linear combination of $T(\mathbf{u}_1),\ldots,T(\mathbf{u}_p)$, and $\{T(\mathbf{u}_1),\ldots,T(\mathbf{u}_p)\}$ spans $T(H)$. By the Spanning Set Theorem, this set contains a basis for $T(H)$. This basis then has not more than p vectors, and $\dim T(H) \leq p = \dim H$.

52. Since H is a nonzero subspace of a finite-dimensional vector space V, H is finite-dimensional and has a basis. Let $\{\mathbf{u}_1,\ldots\mathbf{u}_p\}$ be a basis for H. In Exercise 51 above it was shown that $\{T(\mathbf{u}_1),\ldots,T(\mathbf{u}_p)\}$ spans $T(H)$. In Exercise 42 in Section 4.3, it was shown that $\{T(\mathbf{u}_1),\ldots,T(\mathbf{u}_p)\}$ is linearly independent. Thus $\{T(\mathbf{u}_1),\ldots,T(\mathbf{u}_p)\}$ is a basis for $T(H)$, and $\dim T(H) = p = \dim H$.

53. **a**. To find a basis for $\mathbb{R}^5$ which contains the given vectors, we row reduce

$$\begin{bmatrix} -9 & 9 & 6 & 1 & 0 & 0 & 0 & 0 \\ -7 & 4 & 7 & 0 & 1 & 0 & 0 & 0 \\ 8 & 1 & -8 & 0 & 0 & 1 & 0 & 0 \\ -5 & 6 & 5 & 0 & 0 & 0 & 1 & 0 \\ 7 & -7 & -7 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1/3 & 0 & 0 & 1 & 3/7 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 5/7 \\ 0 & 0 & 1 & -1/3 & 0 & 0 & 0 & -3/7 \\ 0 & 0 & 0 & 0 & 1 & 0 & 3 & 22/7 \\ 0 & 0 & 0 & 0 & 0 & 1 & -9 & -53/7 \end{bmatrix}.$$

The first, second, third, fifth, and sixth columns are pivot columns, so these columns of the original matrix ($\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{e}_2,\mathbf{e}_3\}$) form a basis for $\mathbb{R}^5$:

b. The original vectors are the first k columns of A. Since the set of original vectors is assumed to be linearly independent, these columns of A will be pivot columns and the original set of vectors will be included in the basis. Since the columns of A include all the columns of the identity matrix, Col A = $\mathbb{R}^n$.

54. **a**. The B-coordinate vectors of the vectors in C are the columns of the matrix

$$P = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -3 & 0 & 5 & 0 \\ 0 & 0 & 2 & 0 & -8 & 0 & 18 \\ 0 & 0 & 0 & 4 & 0 & -20 & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 & -48 \\ 0 & 0 & 0 & 0 & 0 & 16 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 32 \end{bmatrix}.$$

The matrix P is invertible because it is triangular with nonzero entries along its main diagonal. Thus its columns are linearly independent. Since the coordinate mapping is an isomorphism, this shows that the vectors in C are linearly independent.

b. We know that dim $H = 7$ because B is a basis for H. Now C is a linearly independent set, and the vectors in C lie in H by the trigonometric identities. Thus by the Basis Theorem, C is a basis for H.

4.6 - Change of Basis

Notes: This section depends heavily on the coordinate systems introduced in Section 4.4. The row reduction algorithm that produces $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ can also be deduced from Exercise 22 in Section 2.2, by row reducing $\left[P_{\mathcal{C}} \mid P_{\mathcal{B}}\right]$. to $\left[I \mid P_{\mathcal{C}}^{-1}P_{\mathcal{B}}\right]$. The change-of-coordinates matrix here is interpreted in Section 5.4 as the matrix of the identity transformation relative to two bases.

1. **a**. Since $\mathbf{b}_1 = 6\mathbf{c}_1 - 2\mathbf{c}_2$ and $\mathbf{b}_2 = 9\mathbf{c}_1 - 4\mathbf{c}_2$, $[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$, $[\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} 9 \\ -4 \end{bmatrix}$, and $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix}$.

b. Since $\mathbf{x} = -3\mathbf{b}_1 + 2\mathbf{b}_2$, $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$ and $[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[x]_{\mathcal{B}} = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$

2. **a**. Since $\mathbf{b}_1 = -\mathbf{c}_1 + 4\mathbf{c}_2$ and $\mathbf{b}_2 = 5\mathbf{c}_1 - 3\mathbf{c}_2$, $[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$, $[\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$, and $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} -1 & 5 \\ 4 & -3 \end{bmatrix}$.

b. Since $\mathbf{x} = 5\mathbf{b}_1 + 3\mathbf{b}_2$, $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ and $[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -1 & 5 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 5 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ 11 \end{bmatrix}$

3. Equation (ii) is satisfied by P for all $\mathbf{x}$ in V.

4. Equation (i) is satisfied by P for all $\mathbf{x}$ in V.

5. **a**. Since $\mathbf{a}_1 = 4\mathbf{b}_1 - \mathbf{b}_2$, $\mathbf{a}_2 = -\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$, and $\mathbf{a}_3 = \mathbf{b}_2 - 2\mathbf{b}_3$, $[\mathbf{a}_1]_{\mathcal{B}} = \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}$, $[\mathbf{a}_2]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$,

$[\mathbf{a}_3]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$, and $\underset{\mathcal{B}\leftarrow\mathcal{A}}{P} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}$.

b. Since $\mathbf{x} = 3\mathbf{a}_1 + 4\mathbf{a}_2 + \mathbf{a}_3$, $[\mathbf{x}]_{\mathcal{A}} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}$ and $[\mathbf{x}]_{\mathcal{B}} = \underset{\mathcal{B}\leftarrow\mathcal{A}}{P} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ 2 \end{bmatrix}$.

6. **a**. Since $\mathbf{f}_1 = 2\mathbf{d}_1 - \mathbf{d}_2 + \mathbf{d}_3$, $\mathbf{f}_2 = 3\mathbf{d}_2 + \mathbf{d}_3$, and $\mathbf{f}_3 = -3\mathbf{d}_1 + 2\mathbf{d}_3$, $[\mathbf{f}_1]_{\mathcal{D}} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $[\mathbf{f}_2]_{\mathcal{D}} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$,

$[\mathbf{f}_3]_{\mathcal{D}} = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$, and $\underset{\mathcal{D}\leftarrow\mathcal{F}}{P} = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}$.

b. Since $\mathbf{x} = \mathbf{f}_1 - 2\mathbf{f}_2 + 2\mathbf{f}_3$, $[\mathbf{x}]_{\mathcal{F}} = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ and $[\mathbf{x}]_{\mathcal{D}} = \underset{\mathcal{D}\leftarrow\mathcal{F}}{P}[\mathbf{x}]_{\mathcal{F}} = \begin{bmatrix} 2 & 0 & -3 \\ -1 & 3 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -4 \\ -7 \\ 3 \end{bmatrix}$.

7. To find $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$, row reduce the matrix $[\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{b}_1 \quad \mathbf{b}_2]$:

$[\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{b}_1 \quad \mathbf{b}_2] \sim \begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & -5 & 2 \end{bmatrix}$. Thus $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}$, and $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}^{-1} = \begin{bmatrix} -2 & 1 \\ -5 & 3 \end{bmatrix}$.

8. To find $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$, row reduce the matrix $[\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{b}_1 \quad \mathbf{b}_2]$: so $[\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{b}_1 \quad \mathbf{b}_2] \sim \begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 4 & 3 \end{bmatrix}$.

Thus $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$, and $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}^{-1} = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}$.

9. To find $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$, row reduce the matrix $[\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{b}_1 \quad \mathbf{b}_2]$:

$[\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{b}_1 \quad \mathbf{b}_2] \sim \begin{bmatrix} 1 & 0 & 9 & -2 \\ 0 & 1 & -4 & 1 \end{bmatrix}$. Thus $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 9 & -2 \\ -4 & 1 \end{bmatrix}$, and $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}^{-1} = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix}$.

10. To find $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$, row reduce the matrix $[\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{b}_1 \quad \mathbf{b}_2]$:

$[\mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{b}_1 \quad \mathbf{b}_2] \sim \begin{bmatrix} 1 & 0 & 8 & 3 \\ 0 & 1 & -5 & -2 \end{bmatrix}$. Thus $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 8 & 3 \\ -5 & -2 \end{bmatrix}$, and $\underset{\mathcal{B}\leftarrow\mathcal{C}}{P} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}^{-1} = \begin{bmatrix} 2 & 3 \\ -5 & -8 \end{bmatrix}$.

11. False. See Theorem 15.

12. True. The columns of $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ are coordinate vectors of the linearly independent set B. See the second paragraph after Theorem 15.

13. True. See the first paragraph in the subsection "Change of Basis in $\mathbb{R}^n$."

14. False. The row reduction is discussed after Example 2. The matrix P obtained there satisfies $[\mathbf{x}]_{\mathcal{C}} = P[\mathbf{x}]_{\mathcal{B}}$

15. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\} = \{1-2t+t^2, 3-5t+4t^2, 2t+3t^2\}$ and let $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\} = \{1, t, t^2\}$. The $\mathcal{C}$-coordinate vectors of $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$ are $[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, [\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}, [\mathbf{b}_3]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$. So

$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{bmatrix}$. Let $\mathbf{x} = -1 + 2t$. Then the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ satisfies

$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$. This system may be solved by row reducing its augmented matrix:

$$\begin{bmatrix} 1 & 3 & 0 & -1 \\ -2 & -5 & 2 & 2 \\ 1 & 4 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \text{so } [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}.$$

16. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\} = \{1-3t^2, 2+t-5t^2, 1+2t\}$ and let $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\} = \{1, t, t^2\}$. The $\mathcal{C}$-coordinate vectors of $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_3$ are $[\mathbf{b}_1]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, [\mathbf{b}_2]_{\mathcal{C}} = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}, [\mathbf{b}_3]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. So $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix}$.

Let $\mathbf{x} = t^2$. Then the coordinate vector $[\mathbf{x}]_{\mathcal{B}}$ satisfies $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{C}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. This system may be solved by row reducing its augmented matrix: $\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ -3 & -5 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$, so $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

and $t^2 = 3(1-3t^2) - 2(2+t-5t^2) + (1+2t)$.

17. **a**. $\mathcal{B}$ is a basis for V
b. the coordinate mapping is a linear transformation
c. the product of a matrix and a vector
d. the coordinate vector of $\mathbf{v}$ relative to $\mathcal{B}$

18. **a**. $[\mathbf{b}_1]_{\mathcal{C}} = Q[\mathbf{b}_1]_{\mathcal{B}} = Q\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} = Q\mathbf{e}_1$

b. $[\mathbf{b}_k]_{\mathcal{C}}$

c. $[\mathbf{b}_k]_{\mathcal{C}} = Q[\mathbf{b}_k]_{\mathcal{B}} = Q\mathbf{e}_k$

19. **a**. Since we found P in Exercise 54 of Section 4.5, we can calculate that

$$P^{-1} = \frac{1}{32}\begin{bmatrix}32 & 0 & 16 & 0 & 12 & 0 & 10\\ 0 & 32 & 0 & 24 & 0 & 20 & 0\\ 0 & 0 & 16 & 0 & 16 & 0 & 15\\ 0 & 0 & 0 & 8 & 0 & 10 & 0\\ 0 & 0 & 0 & 0 & 4 & 0 & 6\\ 0 & 0 & 0 & 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}.$$

b. Since P is the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{B}$, P^{-1} will be the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$. By Theorem 15, the columns of P^{-1} will be the $\mathcal{C}$-coordinate vectors of the basis vectors in $\mathcal{B}$. Thus

$$\cos^2 t = \frac{1}{2}(1+\cos 2t)$$

$$\cos^3 t = \frac{1}{4}(3\cos t+\cos 3t)$$

$$\cos^4 t = \frac{1}{8}(3+4\cos 2t+\cos 4t)$$

$$\cos^5 t = \frac{1}{16}(10\cos t+5\cos 3t+\cos 5t)$$

$$\cos^6 t = \frac{1}{32}(10+15\cos 2t+6\cos 4t+\cos 6t)$$

20. The $\mathcal{C}$-coordinate vector of the integrand is (0, 0, 0, 5, –6, 5, –12). Using P^{-1} from the previous exercise, the $\mathcal{B}$- coordinate vector of the integrand will be

$$P^{-1}(0,0,0,5,-6,5,-12) = (-6, 55/8, -69/8, 45/16, -3, 5/16, -3/8)$$

Thus the integral may be rewritten as

$$\int -6+\frac{55}{8}\cos t-\frac{69}{8}\cos 2t+\frac{45}{16}\cos 3t-3\cos 4t+\frac{5}{16}\cos 5t-\frac{3}{8}\cos 6t\,dt,$$

which equals

$$-6t+\frac{55}{8}\sin t-\frac{69}{16}\sin 2t+\frac{15}{16}\sin 3t-\frac{3}{4}\sin 4t+\frac{1}{16}\sin 5t-\frac{1}{16}\sin 6t+C.$$

21. a. If $\mathcal{C}$ is the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, then the columns of P are $[\mathbf{u}_1]_{\mathcal{C}}$, $[\mathbf{u}_2]_{\mathcal{C}}$, and $[\mathbf{u}_3]_{\mathcal{C}}$. So $\mathbf{u}_j = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3][\mathbf{u}_j]_{\mathcal{C}}$, and $[\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3] = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]P$. In the current exercise,

$$[\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3] = \begin{bmatrix} -2 & -8 & -7 \\ 2 & 5 & 2 \\ 3 & 2 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ -3 & -5 & 0 \\ 4 & 6 & 1 \end{bmatrix} = \begin{bmatrix} -6 & -6 & -5 \\ -5 & -9 & 0 \\ 21 & 32 & 3 \end{bmatrix}.$$

b. Analogously to part a., $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3] = [\mathbf{w}_1 \quad \mathbf{w}_2 \quad \mathbf{w}_3]P$, so $[\mathbf{w}_1 \quad \mathbf{w}_2 \quad \mathbf{w}_3] =$ $[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]P^{-1}$. In the current exercise, $[\mathbf{w}_1 \quad \mathbf{w}_2 \quad \mathbf{w}_3] = \begin{bmatrix} -2 & -8 & -7 \\ 2 & 5 & 2 \\ 3 & 2 & 6 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ -3 & -5 & 0 \\ 4 & 6 & 1 \end{bmatrix}^{-1}$

$$= \begin{bmatrix} -2 & -8 & -7 \\ 2 & 5 & 2 \\ 3 & 2 & 6 \end{bmatrix}\begin{bmatrix} 5 & 8 & 5 \\ -3 & -5 & -3 \\ -2 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 28 & 38 & 21 \\ -9 & -13 & -7 \\ -3 & 2 & 3 \end{bmatrix}.$$

22. a. $\underset{\mathcal{D}\leftarrow\mathcal{B}}{P} = \underset{\mathcal{D}\leftarrow\mathcal{C}}{P}\,\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$

Let $\mathbf{x}$ be any vector in the two-dimensional vector space. Since $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$ is the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{C}$ and $\underset{\mathcal{D}\leftarrow\mathcal{C}}{P}$ is the change-of-coordinates matrix from $\mathcal{C}$ to $\mathcal{D}$, $[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\mathbf{x}]_{\mathcal{B}}$ and $[\mathbf{x}]_{\mathcal{D}} = \underset{\mathcal{D}\leftarrow\mathcal{C}}{P}[\mathbf{x}]_{\mathcal{C}} = \underset{\mathcal{D}\leftarrow\mathcal{C}}{P}\,\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\mathbf{x}]_{\mathcal{B}}$. But since $\underset{\mathcal{D}\leftarrow\mathcal{B}}{P}$ is the change-of-coordinates matrix from $\mathcal{B}$ to $\mathcal{D}$, $[\mathbf{x}]_{\mathcal{D}} = \underset{\mathcal{D}\leftarrow\mathcal{B}}{P}[\mathbf{x}]_{\mathcal{B}}$. Thus $\underset{\mathcal{D}\leftarrow\mathcal{B}}{P}[\mathbf{x}]_{\mathcal{B}} = \underset{\mathcal{D}\leftarrow\mathcal{C}}{P}\,\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}[\mathbf{x}]_{\mathcal{B}}$ for any vector $[\mathbf{x}]_{\mathcal{B}}$ in $\mathbb{R}^2$, and $\underset{\mathcal{D}\leftarrow\mathcal{B}}{P} = \underset{\mathcal{D}\leftarrow\mathcal{C}}{P}\,\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$.

b. For example, let $\mathcal{B} = \left\{\begin{bmatrix} 7 \\ 5 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \end{bmatrix}\right\}$, $\mathcal{C} = \left\{\begin{bmatrix} 1 \\ -5 \end{bmatrix}, \begin{bmatrix} -2 \\ 2 \end{bmatrix}\right\}$, and $\mathcal{D} = \left\{\begin{bmatrix} -1 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ -5 \end{bmatrix}\right\}$. Then we can calculate the change-of-coordinates matrices:

$$\begin{bmatrix} 1 & -2 & 7 & -3 \\ -5 & 2 & 5 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & -5 & 2 \end{bmatrix} \Rightarrow \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 1 & -2 \\ 8 & -5 & -5 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -8/3 \\ 0 & 1 & 1 & -14/3 \end{bmatrix} \Rightarrow \underset{\mathcal{D}\leftarrow\mathcal{C}}{P} = \begin{bmatrix} 0 & -8/3 \\ 1 & -14/3 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 7 & -3 \\ 8 & -5 & 5 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 40/3 & -16/3 \\ 0 & 1 & 61/3 & -25/3 \end{bmatrix} \Rightarrow \underset{\mathcal{D}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 40/3 & -16/3 \\ 61/3 & -25/3 \end{bmatrix}$$

One confirms easily that $\underset{\mathcal{D}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} 40/3 & -16/3 \\ 61/3 & -25/3 \end{bmatrix} = \begin{bmatrix} 0 & -8/3 \\ 1 & -14/3 \end{bmatrix}\begin{bmatrix} -3 & 1 \\ -5 & 2 \end{bmatrix} = \underset{\mathcal{D}\leftarrow\mathcal{C}}{P}\,\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$

4.7 - Digital Signal Processing

Notes When I asked my colleague in electrical engineering for advice on curriculum in linear algebra, he immediately mentioned that signals are a vector space and filters are just linear transformations. The material in this section is foundational for understanding the underlying structure of signals and an opportunity for students to see the power of abstract reasoning. Exercises 1-4 give students the opportunity to practice adding signals. Exercises 5-14 explore properties of LTI. And Exercises 25-32 work with the ideas of subspace and basis.

For Exercises 1-4, the indicated sums are for the signals in Table 1 of the text.

1. $\chi + \alpha = (..., 0, 2, 0, 2, 0, 2, 0, ...)$.

2. $\chi - \alpha = (..., 2, 0, 2, 0, 2, 0, 2, ...)$.

3. $\upsilon + 2\alpha = (..., -2, 2, -2, 3, -1, 3, -1, ...)$.

4. $\upsilon - 3\alpha = (... , 3, -3, 3, -2, 4, -2, 4, ...)$.

For Exercises 5-8, recall that $I(\{x_k\}) = \{x_k\}$ and $S(\{x_k\}) = \{x_{k-1}\}$. Apply the transformation to each signal from the table and observe which signals get mapped to the zero signal.

5. $(I+S)(\alpha) = I(\alpha) + S(\alpha) = \{(-1)^k\} + \{(-1)^{k-1}\} = \{0\}$, and α is the only signal from Table 1 that gets mapped to the zero signal by $I+S$.

6. $(I-S)(\chi) = I(\chi) - S(\chi) = \{1\} - \{1\} = \{0\}$, and χ is the only signal from Table 1 that gets mapped to the zero signal by $I-S$.

7. $(I-cS)(\varepsilon_c) = I(\varepsilon_c) - cS(\varepsilon_c) = \{(c)^k\} - c\{(c)^{k-1}\} = \{0\}$, and ε_c is the only signal from Table 1 that gets mapped to the zero signal by $I-cS$.

8. $(I - S - S^2)(F) = \{x_k - x_{k-1} - x_{k-2}\} = \{0\}$

9. Notice
$T(\{x_k\}+\{y_k\}) = T(\{x_k + y_k\}) = \{ (x_k + y_k) - (x_{k-1} + y_{k-1})\} = \{x_k - x_{k-1}\} + \{y_k - y_{k-1}\} = T(\{x_k\}) + T(\{y_k\})$
and

$T(c\{x_k\}) = T(\{cx_k\}) = \{ (cx_k) - (cx_{k-1})\} = c\{x_k - x_{k-1}\} = cT(\{x_k\})$, hence T is a linear transformation.
Finally, $T(\{x_k\}) = \{x_k - x_{k-1}\}$ and $T(\{x_{k+q}\}) = \{x_{k+q} - x_{k-1+q}\}$, so T is time invariant.

10. Notice

$$M_3(\{x_k\}+\{y_k\}) = M_3(\{x_k + y_k\}) = \left\{ \frac{1}{3}\left((x_{k-2} + y_{k-2}) + (x_{k-1} + y_{k-1}) + (x_k + y_k)\right)\right\}$$

$$= \left\{\frac{1}{3}(x_{k-2} + x_{k-1} + x_k)\right\} + \left\{\frac{1}{3}(y_{k-2} + y_{k-1} + y_k)\right\} = M_3(\{x_k\}) + M_3(\{y_k\}) \text{ and}$$

$$M_3(c\{x_k\}) = M_3(\{cx_k\}) = \left\{ \frac{1}{3}(cx_{k-2} + cx_{k-1} + cx_k)\right\} = c\left\{\frac{1}{3}(x_{k-2} + x_{k-1} + x_k)\right\} = cM_3(\{x_k\}) \text{ and hence}$$

M_3 is a linear transformation. Finally,

$M_3(\{x_k\}) = \left\{\frac{1}{3}(x_{k-2} + x_{k-1} + x_k)\right\}$ and $M_3(\{x_{k+q}\}) = \left\{\frac{1}{3}(x_{k-2+q} + x_{k-1+q} + x_{k+q})\right\}$, so T is time invariant.

11. Since $T(\{x_k\}) = \{x_k - x_{k-1}\}$, any signal $\{x_k\}$ for which $x_k = x_{k-1}$ will be in the kernel of T. For example, $\chi = (\ldots, 1, 1, 1, 1, 1, 1, 1, 1, \ldots)$ is in the kernel of T.

12. Since $M_3(\{x_k\}) = \left\{\frac{1}{3}(x_{k-2} + x_{k-1} + x_k)\right\}$, any signal $\{x_k\}$ for which $x_k = -(x_{k-1} + x_{k-2})$ will be in the kernel of M_3. For example, $(\ldots, 1, 1, -2, 1, 1, -2, 1, 1, \ldots)$ is in the kernel of M_3.

13. Apply T to any signal to get a signal in the range of T. For example, $T(\delta) = (\ldots, 0, 0, 0, 1, -1, 0, 0, \ldots)$, is the range of T.

14. Apply M_3 to any signal to get a signal in the range of M_3. For example,
$M_3(\delta) = \left(\ldots, 0, 0, 0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, \ldots\right)$, is the range of T.

15. True. See Theorem 17.

16. False. See Theorem 18.

17. False. See Theorem 17.

18. False. The vectors in each space are quite different.

19. True. Compare the definition of an LTI to the definition of a linear transformation. See Theorem 16.

20. True. See Example 2.

21. False. The signals are a vector space.

22. False. Review the definition of an LTI.

Guess and check or working backwards through the solution to Practice Problem 3 are two good ways to find solutions to Exercises 23 and 24.

23. Notice $I - \frac{3}{4}S$ has $\{x_k\} = \left\{\left(\frac{3}{4}\right)^k\right\}$ in its kernel. Other answers are possible.

24. Notice $I + \frac{2}{3}S$ has $\{x_k\} = \left\{\left(\frac{-2}{3}\right)^k\right\}$ in its kernel. Other answers are possible.

25. Set $r = 0$ to see that the zero signal is in W. Notice that the sum of two signals is in W, as is any scalar multiple of a signal in W.

26. Set $r = 0$ to see that the zero signal is in *W*. Notice that the sum of two signals is in *W*, as is any scalar multiple of a signal in *W*.

27. A typical element in *W* looks like $r(\ldots,1,0,1,0,1,0,1,\ldots)$, so $\{\chi-\alpha\}$ is a basis for *W*. This subspace has dimension 1.

28. A typical element in *W* looks like $r(\ldots,0,0,0,1,1,1,1,\ldots)$, so $\{\upsilon\}$ is a basis for *W*. This subspace has dimension 1.

29. Set $r_j = 0$ for all *j* to see that the zero signal is in *W*. Notice that the sum of two signals is in *W*, as is any scalar multiple of a signal in *W*.

30. Set $r_j = 0$ for all *j* to see that the zero signal is in *W*. Notice that the sum of two signals is in *W*, as is any scalar multiple of a signal in *W*.

31. A typical element in *W* looks like $\sum_{j=-\infty}^{\infty} r_{2j-1}S^{2j-1}(\delta)$, so $\beta = \{S^{2j-1}(\delta) \mid \text{ with } j \text{ any integer}\}$ is an infinite linearly independent subset of *W*, hence *W* is infinite dimensional. Note that β is not a basis, since not every element in *W* can be generated using a finite sum of elements from β.

32. A typical element in *W* looks like $\sum_{j=0}^{\infty} r_j S^j(\delta)$, so $\beta = \{S^j(\delta) \mid \text{ with } j \text{ any positive integer}\}$ is an infinite linearly independent subset of *W*, hence *W* is infinite dimensional. Note that β is not a basis, since not every element in *W* can be generated using a finite sum of elements from β.

4.8 - Applications to Difference Equations

Notes: This is an important section for engineering students and worth extra class time. To spend only one lecture on this section, you could cover through Example 5, but assign the somewhat lengthy Example 3 for reading. Finding a spanning set for the solution space of a difference equation uses the Basis Theorem (Section 4.5) and Theorem 17 in this section, and demonstrates the power of the theory of Chapter 4 in helping to solve applied problems. This section anticipates Section 5.7 on differential equations. The reduction of an n^{th} order difference equation to a linear system of first order difference equations was introduced in Section 1.10, and is revisited in Sections 5.6 and 5.9. Example 3 is the background for Exercise 34 in Section 6.5.

1. Let $y_k = 2^k$. Then $y_{k+2} + 2y_{k+1} - 8y_k = 2^{k+2} + 2(2^{k+1}) - 8(2^k) = 2^k(2^2 + 2^2 - 8) = 2^k(0) = 0$ for all k

Since the difference equation holds for all *k*, 2^k is a solution.

Let $y_k = (-4)^k$. Then

$y_{k+2} + 2y_{k+1} - 8y_k = (-4)^{k+2} + 2(-4)^{k+1} - 8(-4)^k = (-4)^k((-4)^2 + 2(-4) - 8) = (-4)^k(0) = 0$ for all k

Since the difference equation holds for all *k*, $(-4)^k$ is a solution.

2. Let $y_k = 3^k$. Then

$y_{k+2} - 9y_k = 3^{k+2} - 9(3^k) = 3^k(3^2 - 9) = 3^k(0) = 0$ for all k

Since the difference equation holds for all k, 3^k is a solution.

Let $y_k = (-3)^k$. Then

$y_{k+2} - 9y_k = (-3)^{k+2} - 9(-3)^k = (-3)^k((-3)^2 - 9) = (-3)^k(0) = 0$ for all k

Since the difference equation holds for all k, $(-3)^k$ is a solution.

3. The signals 2^k and $(-4)^k$ are linearly independent because neither is a multiple of the other; that is, there is no scalar c such that $2^k = c(-4)^k$ for all k. By Theorem 17, the solution set H of the difference equation $y_{k+2} + 2y_{k+1} - 8y_k = 0$ is two-dimensional. By the Basis Theorem, the two linearly independent signals 2^k and $(-4)^k$ form a basis for H.

4. The signals 3^k and $(-3)^k$ are linearly independent because neither is a multiple of the other; that is, there is no scalar c such that $3^k = c(-3)^k$ for all k. By Theorem 17, the solution set H of the difference equation $y_{k+2} - 9y_k = 0$ is two-dimensional. By the Basis Theorem, the two linearly independent signals 3^k and $(-3)^k$ form a basis for H.

5. Let $y_k = (-3)^k$. Then

$y_{k+2} + 6y_{k+1} + 9y_k = (-3)^{k+2} + 6(-3)^{k+1} + 9(-3)^k = (-3)^k((-3)^2 + 6(-3) + 9) = (-3)^k(0) = 0$ for all k

Since the difference equation holds for all k, $(-3)^k$ is in the solution set H.

Let $y_k = k(-3)^k$. Then

$y_{k+2} + 6y_{k+1} + 9y_k = (k+2)(-3)^{k+2} + 6(k+1)(-3)^{k+1} + 9k(-3)^k$

$= (-3)^k((k+2)(-3)^2 + 6(k+1)(-3) + 9k)$

$= (-3)^k(9k + 18 - 18k - 18 + 9k) = (-3)^k(0) = 0$ for all k

Since the difference equation holds for all k, $k(-3)^k$ is in the solution set H.

The signals $(-3)^k$ and $k(-3)^k$ are linearly independent because neither is a multiple of the other; that is, there is no scalar c such that $(-3)^k = ck(-3)^k$ for all k and there is no scalar c such that $c(-3)^k = k(-3)^k$ for all k . By Theorem 17, dim H = 2, so the two linearly independent signals 3^k and $(-3)^k$ form a basis for H by the Basis Theorem.

6. Let $y_k = 5^k \cos\frac{k\pi}{2}$. Then

$$y_{k+2} + 25y_k = 5^{k+2}\cos\frac{(k+2)\pi}{2} + 25\left(5^k\cos\frac{k\pi}{2}\right) = 5^k\left(5^2\cos\frac{(k+2)\pi}{2} + 25\cos\frac{k\pi}{2}\right)$$

$$= 25\cdot 5^k\left(\cos\left(\frac{k\pi}{2} + \pi\right) + \cos\frac{k\pi}{2}\right) = 25\cdot 5^k(0) = 0 \text{ for all } k$$

since $\cos(t + \pi) = -\cos t$ for all t. Since the difference equation holds for all k, $5^k\cos\frac{k\pi}{2}$ is in the solution set H.

Let $y_k = 5^k \sin\frac{k\pi}{2}$. Then

$$y_{k+2} + 25y_k = 5^{k+2}\sin\frac{(k+2)\pi}{2} + 25\left(5^k \sin\frac{k\pi}{2}\right) = 5^k\left(5^2\sin\frac{(k+2)\pi}{2} + 25\sin\frac{k\pi}{2}\right)$$

$$= 25\cdot 5^k\left(\sin\left(\frac{k\pi}{2}+\pi\right)+\sin\frac{k\pi}{2}\right) = 25\cdot 5^k(0) = 0 \text{ for all } k$$

since $\sin(t+\pi) = -\sin t$ for all t. Since the difference equation holds for all k, $5^k \sin\frac{k\pi}{2}$ is in the solution set H.

The signals $5^k \cos\frac{k\pi}{2}$ and $5^k \sin\frac{k\pi}{2}$ are linearly independent because neither is a multiple of the other. By Theorem 17, dim $H = 2$, so the two linearly independent signals $5^k \cos\frac{k\pi}{2}$ and $5^k \sin\frac{k\pi}{2}$ form a basis for H by the Basis Theorem.

7. Compute and row reduce the Casorati matrix for the signals 1^k, 2^k, and $(-2)^k$, setting $k = 0$ for convenience:

$$\begin{bmatrix} 1^0 & 2^0 & (-2)^0 \\ 1^1 & 2^1 & (-2)^1 \\ 1^2 & 2^2 & (-2)^2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This Casorati matrix is row equivalent to the identity matrix, thus is invertible by the IMT. Hence the set of signals $\{1^k, 2^k, (-2)^k\}$ is linearly independent in $\mathbb{S}$. The exercise states that these signals are in the solution set H of a third-order difference equation. By Theorem 17, dim $H = 3$, so the three linearly independent signals 1^k, 2^k, $(-2)^k$ form a basis for H by the Basis Theorem.

8. Compute and row reduce the Casorati matrix for the signals 2^k, 4^k, and $(-5)^k$, setting $k = 0$ for convenience: $\begin{bmatrix} 2^0 & 4^0 & (-5)^0 \\ 2^1 & 4^1 & (-5)^1 \\ 2^2 & 4^2 & (-5)^2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. This Casorati matrix is row equivalent to the identity matrix, thus is invertible by the IMT. Hence the set of signals $\{2^k, 4^k, (-5)^k\}$ is linearly independent in $\mathbb{S}$. The exercise states that these signals are in the solution set H of a third-order difference equation. By Theorem 17, dim $H = 3$, so the three linearly independent signals 2^k, 4^k, $(-5)^k$ form a basis for H by the Basis Theorem.

9. Compute and row reduce the Casorati matrix for the signals 1^k, $3^k \cos\frac{k\pi}{2}$, and $3^k \sin\frac{k\pi}{2}$, setting $k =$ 0 for convenience: $\begin{bmatrix} 1^0 & 3^0\cos 0 & 3^0\sin 0 \\ 1^1 & 3^1\cos\frac{\pi}{2} & 3^1\sin\frac{\pi}{2} \\ 1^2 & 3^2\cos\pi & 3^2\sin\pi \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. This Casorati matrix is row equivalent to the identity matrix, thus is invertible by the IMT. Hence the set of signals $\{1^k, 3^k\cos\frac{k\pi}{2}, 3^k\sin\frac{k\pi}{2}\}$ is linearly independent in $\mathbb{S}$. The exercise states that these signals are in the solution set H of a third-order difference equation. By Theorem 17, dim $H = 3$, so the three linearly independent signals 1^k, $3^k \cos\frac{k\pi}{2}$, and $3^k \sin\frac{k\pi}{2}$, form a basis for *H* by the Basis Theorem.

10. Compute and row reduce the Casorati matrix for the signals $(-1)^k$, $k(-1)^k$, and 5^k, setting $k = 0$ for convenience: $\begin{bmatrix} (-1)^0 & 0(-1)^0 & 5^0 \\ (-1)^1 & 1(-1)^1 & 5^1 \\ (-1)^2 & 2(-1)^2 & 5^2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. This Casorati matrix is row equivalent to the identity matrix, thus is invertible by the IMT. Hence the set of signals $\{(-1)^k, k(-1)^k, 5^k\}$ is linearly independent in $\mathbb{S}$. The exercise states that these signals are in the solution set *H* of a third-order difference equation. By Theorem 17, dim $H = 3$, so the three linearly independent signals $(-1)^k$, $k(-1)^k$, and 5^k form a basis for *H* by the Basis Theorem.

11. The solution set *H* of this third-order difference equation has dim $H = 3$ by Theorem 17. The two signals $(-1)^k$ and 3^k cannot possibly span a three-dimensional space, and so cannot be a basis for *H*.

12. The solution set *H* of this fourth-order difference equation has dim $H = 4$ by Theorem 17. The two signals 1^k and $(-1)^k$ cannot possibly span a four-dimensional space, and so cannot be a basis for *H*.

13. The auxiliary equation for this difference equation is $r^2 - r + 2/9 = 0$. By the quadratic formula (or factoring), $r = 2/3$ or $r = 1/3$, so two solutions of the difference equation are $(2/3)^k$ and $(1/3)^k$. The signals $(2/3)^k$ and $(1/3)^k$ are linearly independent because neither is a multiple of the other. By Theorem 17, the solution space is two-dimensional, so the two linearly independent signals $(2/3)^k$ and $(1/3)^k$ form a basis for the solution space by the Basis Theorem.

14. The auxiliary equation for this difference equation is $r^2 - 7r + 12 = 0$. By the quadratic formula (or factoring), $r = 3$ or $r = 4$, so two solutions of the difference equation are 3^k and 4^k. The signals 3^k and 4^k are linearly independent because neither is a multiple of the other. By Theorem 17, the solution space is two-dimensional, so the two linearly independent signals 3^k and 4^k form a basis for the solution space by the Basis Theorem.

15. The auxiliary equation for this difference equation is $r^2 - 25 = 0$. By the quadratic formula (or factoring), $r = 5$ or $r = -5$, so two solutions of the difference equation are 5^k and $(-5)^k$. The signals 5^k and $(-5)^k$ are linearly independent because neither is a multiple of the other. By Theorem 17, the solution space is two-dimensional, so the two linearly independent signals 5^k and $(-5)^k$ form a basis for the solution space by the Basis Theorem.

16. The auxiliary equation for this difference equation is $16r^2 + 8r - 3 = 0$. By the quadratic formula (or factoring), $r = 1/4$ or $r = -3/4$, so two solutions of the difference equation are $(1/4)^k$ and $(-3/4)^k$. The signals $(1/4)^k$ and $(-3/4)^k$ are linearly independent because neither is a multiple of the other. By Theorem 17, the solution space is two-dimensional, so the two linearly independent signals $(1/4)^k$ and $(-3/4)^k$ form a basis for the solution space by the Basis Theorem.

17. To find the general solution of the original equation, we look at the auxiliary equation:

$$r^2 - r - 1 = 0$$

Using the quadratic formula we find the roots are $r = \dfrac{1 \pm \sqrt{5}}{2}$

and the general solution is $y_k = c_1\left(\dfrac{1+\sqrt{5}}{2}\right)^k + c_2\left(\dfrac{1-\sqrt{5}}{2}\right)^k$.

We can then use the initial conditions $y_0 = 0$ and $y_1 = 1$ to evaluate c_1 and c_2.

When $k = 0$: $y_0 = c_1\left(\frac{1+\sqrt{5}}{2}\right)^0 + c_2\left(\frac{1-\sqrt{5}}{2}\right)^0 = 0 \Rightarrow c_1 = -c_2$

When $k = 1$: $y_1 = c_1\left(\frac{1+\sqrt{5}}{2}\right)^1 + c_2\left(\frac{1-\sqrt{5}}{2}\right)^1 = 1$

Together these imply, $c_1 = \dfrac{1}{\sqrt{5}}$ and $c_2 = -\dfrac{1}{\sqrt{5}}$.

So the general solution of the Fibonacci sequence is $y_k = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^k$.

18. With the new initial conditions, $y_0 = 1, y_1 = 2, y_2 = 3, y_3 = 5, y_4 = 8, y_5 = 13$. We can then use the initial conditions $y_0 = 1$ and $y_1 = 2$ to evaluate c_1 and c_2.

When $k = 0$: $y_0 = c_1\left(\frac{1+\sqrt{5}}{2}\right)^0 + c_2\left(\frac{1-\sqrt{5}}{2}\right)^0 = 1 \Rightarrow c_2 = 1 - c_1$

When $k = 1$: $y_1 = c_1\left(\frac{1+\sqrt{5}}{2}\right)^1 + (1 - c_1)\left(\frac{1-\sqrt{5}}{2}\right)^1 = 2$

Together these imply, $c_1 = \frac{3+\sqrt{5}}{2\sqrt{5}}$ and $c_2 = \frac{-3+\sqrt{5}}{2\sqrt{5}}$.

So the general solution of the Fibonacci sequence is $y_k = \left(\frac{3+\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1+\sqrt{5}}{2\sqrt{5}}\right)^k + \left(\frac{-3+\sqrt{5}}{2\sqrt{5}}\right)\left(\frac{1-\sqrt{5}}{2\sqrt{5}}\right)^k$

19. Letting $a = .9$ and $b = 4/9$ gives the difference equation $Y_{k+2} - 1.3Y_{k+1} + .4Y_k = 1$. First we find a particular solution $Y_k = T$ of this equation, where T is a constant. The solution of the equation $T - 1.3T + .4T = 1$ is $T = 10$, so 10 is a particular solution to $Y_{k+2} - 1.3Y_{k+1} + .4Y_k = 1$. Next we solve the homogeneous difference equation $Y_{k+2} - 1.3Y_{k+1} + .4Y_k = 0$. The auxiliary equation for this difference equation is $r^2 - 1.3r + .4 = 0$. By the quadratic formula (or factoring), $r = .8$ or $r = .5$, so two solutions of the homogeneous difference equation are $.8^k$ and $.5^k$. The signals $(.8)^k$ and $(.5)^k$ are linearly independent because neither is a multiple of the other. By Theorem 17, the solution space is two-dimensional, so the two linearly independent signals $(.8)^k$ and $(.5)^k$ form a basis for the solution space of the homogeneous difference equation by the Basis Theorem. Translating the solution space of the homogeneous difference equation by the particular solution 10 of the nonhomogeneous difference equation gives us the general solution of $Y_{k+2} - 1.3Y_{k+1} + .4Y_k = 1$:

$Y_k = c_1(.8)^k + c_2(.5)^k + 10$. As k increases the first two terms in the solution approach 0, so Y_k approaches 10.

20. Letting $a = .9$ and $b = .5$ gives the difference equation $Y_{k+2} - 1.35Y_{k+1} + .45Y_k = 1$. First we find a particular solution $Y_k = T$ of this equation, where T is a constant. The solution of the equation $T - 1.35T + .45T = 1$ is $T = 10$, so 10 is a particular solution to $Y_{k+2} - 1.35Y_{k+1} + .45Y_k = 1$. Next we solve the homogeneous difference equation $Y_{k+2} - 1.35Y_{k+1} + .45Y_k = 0$. The auxiliary equation for this difference equation is $r^2 - 1.35r + .45 = 0$. By the quadratic formula (or factoring), $r = .6$ or $r = .75$, so two solutions of the homogeneous difference equation are $.6^k$ and $.75^k$. The signals $(.6)^k$ and $(.75)^k$ are linearly independent because neither is a multiple of the other. By Theorem 17, the solution space is two-dimensional, so the two linearly independent signals $(.6)^k$ and $(.75)^k$ form a basis for the solution space of the homogeneous difference equation by the Basis Theorem. Translating the solution space of the homogeneous difference equation by the particular solution 10 of the nonhomogeneous difference equation gives us the general solution of $Y_{k+2} - 1.35Y_{k+1} + .45Y_k = 1$: $Y_k = c_1(.6)^k + c_2(.75)^k + 10$.

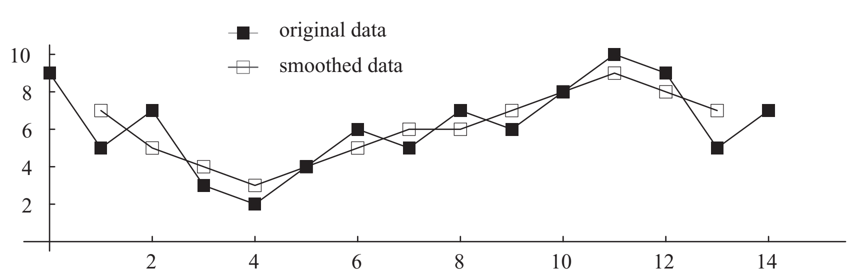
21. The auxiliary equation for this difference equation is $r^2 + 4r + 1 = 0$. By the quadratic formula, $r = -2 + \sqrt{3}$ or $r = -2 - \sqrt{3}$, so two solutions of the difference equation are $(-2+\sqrt{3})^k$ and $(-2-\sqrt{3})^k$. The signals $(-2+\sqrt{3})^k$ and $(-2-\sqrt{3})^k$ are linearly independent because neither is a multiple of the other. By Theorem 17, the solution space is two-dimensional, so the two linearly independent signals $(-2+\sqrt{3})^k$ and $(-2-\sqrt{3})^k$ form a basis for the solution space by the Basis Theorem. Thus a general solution to this difference equation is $y_k = c_1(-2+\sqrt{3})^k + c_2(-2-\sqrt{3})^k$.

22. Let $a = -2 + \sqrt{3}$ and $b = -2 - \sqrt{3}$. Using the solution from the previous exercise, we find that $y_1 = c_1 a + c_2 b = 5000$ and $y_N = c_1 a^N + c_2 b^N = 0$. This is a system of linear equations with variables c_1 and c_2 whose augmented matrix may be row reduced: $\begin{bmatrix} a & b & 5000 \\ a^N & b^N & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \dfrac{5000b^N}{b^N a - a^N b} \\ 0 & 1 & \dfrac{5000a^N}{b^N a - a^N b} \end{bmatrix}$

So $c_1 = \dfrac{5000b^N}{b^N a - a^N b}, c_2 = \dfrac{5000a^N}{b^N a - a^N b}$. (Alternatively, Cramer's Rule may be applied to get the same solution). Thus

$$y_k = c_1 a^k + c_2 b^k = \frac{5000(a^k b^N - a^N b^k)}{b^N a - a^N b}$$

23. The smoothed signal z_k has the following values: $z_1 = (9+5+7)/3 = 7$, $z_2 = (5+7+3)/3 = 5$, $z_3 = (7+3+2)/3 = 4$, $z_4 = (3+2+4)/3 = 3$, $z_5 = (2+4+6)/3 = 4$, $z_6 = (4+6+5)/3 = 5$, $z_7 = (6+5+7)/3 = 6$, $z_8 = (5+7+6)/3 = 6$, $z_9 = (7+6+8)/3 = 7$, $z_{10} = (6+8+10)/3 = 8$, $z_{11} = (8+10+9)/3 = 9$, $z_{12} = (10+9+5)/3 = 8$, $z_{13} = (9+5+7)/3 = 7$.



24. **a**. The smoothed signal z_k has the following values:

$$z_0 = .35y_2 + .5y_1 + .35y_0 = .35(0) + .5(.7) + .35(3) = 1.4,$$
$$z_1 = .35y_3 + .5y_2 + .35y_1 = .35(-.7) + .5(0) + .35(.7) = 0,$$
$$z_2 = .35y_4 + .5y_3 + .35y_2 = .35(-.3) + .5(-.7) + .35(0) = -1.4,$$
$$z_3 = .35y_5 + .5y_4 + .35y_3 = .35(-.7) + .5(-.3) + .35(-.7) = -2,$$
$$z_4 = .35y_6 + .5y_5 + .35y_4 = .35(0) + .5(-.7) + .35(-.3) = -1.4,$$
$$z_5 = .35y_7 + .5y_6 + .35y_5 = .35(.7) + .5(0) + .35(-.7) = 0,$$
$$z_6 = .35y_8 + .5y_7 + .35y_6 = .35(3) + .5(.7) + .35(0) = 1.4,$$
$$z_7 = .35y_9 + .5y_8 + .35y_7 = .35(.7) + .5(3) + .35(.7) = 2,$$
$$z_8 = .35y_{10} + .5y_9 + .35y_8 = .35(0) + .5(.7) + .35(3) = 1.4, \ldots$$

b. This signal is two times the signal output by the filter when the input (in Example 2) was $y = \cos(\pi t/4)$. This is expected because the filter is linear. The output from the input $2\cos(\pi t/4) + \cos(3\pi t/4)$ should be two times the output from $\cos(\pi t/4)$ plus the output from $\cos(3\pi t/4)$ (which is zero).

25. **a**. $y_{k+1} - 1.01y_k = -450,\ \ y_0 = 10{,}000.$

b. MATLAB code to create the table:

```
pay = 450, y = 10000, m = 0, table = [0;y]
while y>450
    y = 1.01*y-pay
    m = m+1
    table = [table [m;y]]
end
m,y
```

Mathematica code to create the table:

```
pay = 450; y = 10000; m = 0; balancetable = {{0, y}};
While[y > 450, {y = 1.01*y - pay; m = m + 1,
    AppendTo[balancetable, {m, y}]}];
m
y
```

c. At month 26, the last payment is \$114.88. The total paid by the borrower is \$11,364.88.

26. **a**. $y_{k+1} - 1.005y_k = 200,\ \ y_0 = 1{,}000.$

b. MATLAB code to create the table:

```
pay = 200, y = 1000, m = 0, table = [0;y]
for m = 1: 60
    y = 1.005*y+pay
    table = [table [m;y]]
end
```

```
interest = y-60*pay-1000
```

Mathematica code to create the table:

```
pay = 200; y = 1000; amounttable = {{0, y}};
Do[{y = 1.005*y + pay;
    AppendTo[amounttable, {m, y}]},{m,1,60}];
interest = y-60*pay-1000
```

c. The total is \$6213.55 at $k = 24$, \$12,090.06 at $k = 48$, and \$15,302.86 at $k = 60$. When $k = 60$, the interest earned is \$2302.86.

27. To show that $y_k = k^2$ is a solution of $y_{k+2} + 3y_{k+1} - 4y_k = 10k + 7$, substitute $y_k = k^2$, $y_{k+1} = (k+1)^2$, and $y_{k+2} = (k+2)^2$:

$y_{k+2} + 3y_{k+1} - 4y_k = (k+2)^2 + 3(k+1)^2 - 4k^2 = (k^2 + 4k + 4) + 3(k^2 + 2k + 1) - 4k^2$

$= k^2 + 4k + 4 + 3k^2 + 6k + 3 - 4k^2 = 10k + 7$ for all k

The auxiliary equation for the homogeneous difference equation $y_{k+2} + 3y_{k+1} - 4y_k = 0$ is $r^2 + 3r - 4 = 0$. By the quadratic formula (or factoring), $r = -4$ or $r = 1$, so two solutions of the difference equation are $(-4)^k$ and 1^k. The signals $(-4)^k$ and 1^k are linearly independent because neither is a multiple of the other. By Theorem 17, the solution space is two-dimensional, so the two linearly independent signals $(-4)^k$ and 1^k form a basis for the solution space of the homogeneous difference equation by the Basis Theorem. The general solution to the homogeneous difference equation is thus $c_1(-4)^k + c_2 \cdot 1^k = c_1(-4)^k + c_2$. Adding the particular solution k^2 of the nonhomogeneous difference equation, we find that the general solution of the difference equation $y_{k+2} + 3y_{k+1} - 4y_k = 10k + 7$ is $y_k = k^2 + c_1(-4)^k + c_2$.

28. To show that $y_k = 1 + k$ is a solution of $y_{k+2} - 8y_{k+1} + 15y_k = 8k + 2$, substitute $y_k = 1 + k$, $y_{k+1} = 1 + (k+1) = 2 + k$, and $y_{k+2} = 1 + (k+2) = 3 + k$:

$y_{k+2} - 8y_{k+1} + 15y_k = (3+k) - 8(2+k) + 15(1+k) = 3 + k - 16 - 8k + 15 + 15k = 8k + 2$ for all k

The auxiliary equation for the homogeneous difference equation $y_{k+2} - 8y_{k+1} + 15y_k = 0$ is $r^2 - 8r + 15 = 0$. By the quadratic formula (or factoring), $r = 5$ or $r = 3$, so two solutions of the difference equation are 5^k and 3^k. The signals 5^k and 3^k are linearly independent because neither is a multiple of the other. By Theorem 17, the solution space is two-dimensional, so the two linearly independent signals 5^k and 3^k form a basis for the solution space of the homogeneous difference equation by the Basis Theorem. The general solution to the homogeneous difference equation is thus $c_1 \cdot 5^k + c_2 \cdot 3^k$. Adding the particular solution $1 + k$ of the nonhomogeneous difference equation, we find that the general solution of the difference equation $y_{k+2} - 8y_{k+1} + 15y_k = 8k + 2$ is $y_k = 1 + k + c_1 \cdot 5^k + c_2 \cdot 3^k$.

29. To show that $y_k = 2 - 2k$ is a solution of $y_{k+2} - (9/2)y_{k+1} + 2y_k = 3k + 2$, substitute $y_k = 2 - 2k$, $y_{k+1} = 2 - 2(k+1) = -2k$, and $y_{k+2} = 2 - 2(k+2) = -2 - 2k$:

$y_{k+2} - (9/2)y_{k+1} + 2y_k = (-2 - 2k) - (9/2)(-2k) + 2(2 - 2k) = -2 - 2k + 9k + 4 - 4k = 3k + 2$ for all k

The auxiliary equation for the homogeneous difference equation $y_{k+2}-(9/2)y_{k+1}+2y_k=0$ is $r^2-(9/2)r+2=0$. By the quadratic formula (or factoring), $r = 4$ or $r = 1/2$, so two solutions of the difference equation are 4^k and $(1/2)^k$. The signals 4^k and $(1/2)^k$ are linearly independent because neither is a multiple of the other. By Theorem 17, the solution space is two-dimensional, so the two linearly independent signals 4^k and $(1/2)^k$ form a basis for the solution space of the homogeneous difference equation by the Basis Theorem. The general solution to the homogeneous difference equation is thus $c_1\cdot 4^k+c_2\cdot(1/2)^k=c_1\cdot 4^k+c_2\cdot 2^{-k}$. Adding the particular solution $2-2k$ of the nonhomogeneous difference equation, we find that the general solution of the difference equation $y_{k+2}-(9/2)y_{k+1}+2y_k=3k+2$ is $y_k=2-2k+c_1\cdot 4^k+c_2\cdot 2^{-k}$.

30. To show that $y_k=2k-4$ is a solution of $y_{k+2}+(3/2)y_{k+1}-y_k=1+3k$, substitute $y_k=2k-4$, $y_{k+1}=2(k+1)-4=2k-2$, and $y_{k+2}=2(k+2)-4=2k$:

$y_{k+2}+(3/2)y_{k+1}-y_k=2k+(3/2)(2k-2)-(2k-4)=2k+3k-3-2k+4=1+3k$ for all k

The auxiliary equation for the homogeneous difference equation $y_{k+2}+(3/2)y_{k+1}-y_k=0$ is $r^2+(3/2)r-1=0$. By the quadratic formula (or factoring), $r = -2$ or $r = 1/2$, so two solutions of the difference equation are $(-2)^k$ and $(1/2)^k$. The signals $(-2)^k$ and $(1/2)^k$ are linearly independent because neither is a multiple of the other. By Theorem 17, the solution space is two-dimensional, so the two linearly independent signals $(-2)^k$ and $(1/2)^k$ form a basis for the solution space of the homogeneous difference equation by the Basis Theorem. The general solution to the homogeneous difference equation is thus $c_1\cdot(-2)^k+c_2\cdot(1/2)^k=c_1\cdot(-2)^k+c_2\cdot 2^{-k}$. Adding the particular solution $2k-4$ of the nonhomogeneous difference equation, we find that the general solution of the difference equation $y_{k+2}+(3/2)y_{k+1}-y_k=1+3k$ is $y_k=2k-4+c_1\cdot(-2)^k+c_2\cdot 2^{-k}$.

31. Let $\mathbf{x}_k=\begin{bmatrix}y_k\\y_{k+1}\\y_{k+2}\\y_{k+3}\end{bmatrix}$. Then $\mathbf{x}_{k+1}=\begin{bmatrix}y_{k+1}\\y_{k+2}\\y_{k+3}\\y_{k+4}\end{bmatrix}=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\9&-6&-8&6\end{bmatrix}\begin{bmatrix}y_k\\y_{k+1}\\y_{k+2}\\y_{k+3}\end{bmatrix}=A\mathbf{x}_k$.

32. Let $\mathbf{x}_k=\begin{bmatrix}y_k\\y_{k+1}\\y_{k+2}\end{bmatrix}$. Then $\mathbf{x}_{k+1}=\begin{bmatrix}y_{k+1}\\y_{k+2}\\y_{k+3}\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\-1/16&0&3/4\end{bmatrix}\begin{bmatrix}y_k\\y_{k+1}\\y_{k+2}\end{bmatrix}=A\mathbf{x}_k$.

33. The difference equation is of order 2. Since the equation $y_{k+3}+5y_{k+2}+6y_{k+1}=0$ holds for all k, it holds if k is replaced by $k-1$. Performing this replacement transforms the equation into $y_{k+2}+5y_{k+1}+6y_k=0$, which is also true for all k. The transformed equation has order 2.

34. The order of the difference equation depends on the values of a_1, a_2, and a_3. If $a_3\neq 0$, then the order is 3. If $a_3=0$ and $a_2\neq 0$, then the order is 2. If $a_3=a_2=0$ and $a_1\neq 0$, then the order is 1. If $a_3=a_2=a_1=0$, then the order is 0, and the equation has only the zero signal for a solution.

35. The Casorati matrix $C(k)$ is $C(k)=\begin{bmatrix}y_k&z_k\\y_{k+1}&z_{k+1}\end{bmatrix}=\begin{bmatrix}k^2&2k\,|\,k\,|\\(k+1)^2&2(k+1)\,|\,k+1\,|\end{bmatrix}$. In particular,

$C(0)=\begin{bmatrix}0&0\\1&2\end{bmatrix}, C(-1)=\begin{bmatrix}1&-2\\0&0\end{bmatrix}$, and $C(-2)=\begin{bmatrix}4&-8\\1&-2\end{bmatrix}$, none of which are invertible. In fact, $C(k)$ is not invertible for all k, since

$\det C(k)=2k^2(k+1)|k+1|-2(k+1)^2k|k|=2k(k+1)\big(k|k+1|-(k+1)|k|\big)$

If $k = 0$ or $k = -1$, $\det C(k) = 0$. If $k > 0$, then $k + 1 > 0$ and $k|k+1| - (k+1)|k| = k(k+1) - (k+1)k = 0$, so $\det C(k) = 0$. If $k < -1$, then $k + 1 < 0$ and $k|k+1| - (k+1)|k| = -k(k+1) + (k+1)k = 0$, so $\det C(k) = 0$. Thus $\det C(k)=0$ for all k, and $C(k)$ is not invertible for all k. Since $C(k)$ is not invertible for all k, it provides no information about whether the signals $\{y_k\}$ and $\{z_k\}$ are linearly dependent or linearly independent. In fact, neither signal is a multiple of the other, so the signals $\{y_k\}$ and $\{z_k\}$ are linearly independent.

36. No, the signals could be linearly dependent, since the vector space V of functions considered on the entire real line is not the vector space $\mathbb{S}$ of signals. For example, consider the functions $f(t) = \sin \pi t$, $g(t) = \sin 2\pi t$, and $h(t) = \sin 3\pi t$. The functions f, g, and h are linearly independent in V since they have different periods and thus no function could be a linear combination of the other two. However, sampling the functions at any integer n gives $f(n) = g(n) = h(n) = 0$, so the signals are linearly dependent in $\mathbb{S}$.

Chapter 4 - Supplementary Exercises

1. True. This set is $\text{Span}\{\mathbf{v}_1,\dots \mathbf{v}_p\}$, and every subspace is itself a vector space.

2. True. Any linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_{p-1}$ is also a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_{p-1}, \mathbf{v}_p$ using the zero weight on $\mathbf{v}_p$.

3. False. Counterexample: Take $\mathbf{v}_p = 2\mathbf{v}_1$. Then $\{\mathbf{v}_1,\dots \mathbf{v}_p\}$ is linearly dependent.

4. False. Counterexample: Let $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ be the standard basis for $\mathbb{R}^3$. Then $\{\mathbf{e}_1,\mathbf{e}_2\}$ is a linearly independent set but is not a basis for $\mathbb{R}^3$.

5. True. See the Spanning Set Theorem (Section 4.3).

6. True. By the Basis Theorem, S is a basis for V because S spans V and has exactly p elements. So S must be linearly independent.

7. False. The plane must pass through the origin to be a subspace.

8. False. Counterexample: $\begin{bmatrix}2&5&-2&0\\0&0&7&3\\0&0&0&0\end{bmatrix}$.

9. True. This statement appears before Theorem 7 in Section 4.5.

10. False. Row operations on A do not change the solutions of $A\mathbf{x} = \mathbf{0}$.

11. False. Counterexample: $A=\begin{bmatrix}1&2\\3&6\end{bmatrix}$; A has two nonzero rows but the rank of A is 1.

12. False. If U has k nonzero rows, then rank $A = k$ and $\dim\text{Nul } A = n - k$ by the Rank Theorem.

13. True. Row equivalent matrices have the same number of pivot columns.

14. False. The nonzero rows of A span Row A but they may not be linearly independent.

15. True. The nonzero rows of the reduced echelon form E form a basis for the row space of each matrix that is row equivalent to E.

16. True. If H is the zero subspace, let A be the 3×3 zero matrix. If dim $H = 1$, let $\{\mathbf{v}\}$ be a basis for H and set $A = [\mathbf{v} \quad \mathbf{v} \quad \mathbf{v}]$. If dim $H = 2$, let $\{\mathbf{u},\mathbf{v}\}$ be a basis for H and set $A = [\mathbf{u} \quad \mathbf{v} \quad \mathbf{v}]$, for example. If dim $H = 3$, then $H = \mathbb{R}^3$, so A can be any 3×3 invertible matrix. Or, let $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ be a basis for H and set $A = [\mathbf{u} \quad \mathbf{v} \quad \mathbf{w}]$.

17. False. Counterexample: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. If rank $A = n$ (the number of *columns* in A), then the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

18. True. If $\mathbf{x} \mapsto A\mathbf{x}$ is onto, then Col $A = \mathbb{R}^m$ and rank $A = m$. See Theorem 12(a) in Section 1.9.

19. True. See the second paragraph after Theorem 15 in Section 4.6.

20. The set is Span$S$, where $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -8 \\ 7 \\ 1 \end{bmatrix} \right\}$. Note that S is a linearly dependent set, but each pair of vectors in S forms a linearly independent set. Thus any two of the three vectors $\begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -8 \\ 7 \\ 1 \end{bmatrix}$ will be a basis for Span$S$.

21. The vector $\mathbf{b}$ will be in $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ if and only if there exist constants c_1 and c_2 with $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 = \mathbf{b}$. Row reducing the augmented matrix gives $\begin{bmatrix} -2 & 1 & b_1 \\ 4 & 2 & b_2 \\ -6 & -5 & b_3 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & b_1 \\ 0 & 4 & 2b_1 + b_2 \\ 0 & 0 & b_1 + 2b_2 + b_3 \end{bmatrix}$ so $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ is the set of all (b_1, b_2, b_3) satisfying $b_1 + 2b_2 + b_3 = 0$.

22. The vector $\mathbf{g}$ is not a scalar multiple of the vector $\mathbf{f}$, and $\mathbf{f}$ is not a scalar multiple of $\mathbf{g}$, so the set $\{\mathbf{f}, \mathbf{g}\}$ is linearly independent. Even though the *number* $\mathbf{g}(t)$ is a scalar multiple of $\mathbf{f}(t)$ for each t, the scalar depends on t.

23. The vector $\mathbf{p}_1$ is not zero, and $\mathbf{p}_2$ is not a multiple of $\mathbf{p}_1$. However, $\mathbf{p}_3$ is $2\mathbf{p}_1 + 2\mathbf{p}_2$, so $\mathbf{p}_3$ is discarded. The vector $\mathbf{p}_4$ cannot be a linear combination of $\mathbf{p}_1$ and $\mathbf{p}_2$ since $\mathbf{p}_4$ involves t^2 but $\mathbf{p}_1$ and $\mathbf{p}_2$ do not involve t^2. The vector $\mathbf{p}_5$ is $(3/2)\mathbf{p}_1 - (1/2)\mathbf{p}_2 + \mathbf{p}_4$ (which may not be so easy to see at first.) Thus $\mathbf{p}_5$ is a linear combination of $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_4$, so $\mathbf{p}_5$ is discarded. So the resulting basis is $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_4\}$.

24. Find two polynomials from the set $\{\mathbf{p}_1, \dots, \mathbf{p}_4\}$ that are not multiples of one another. This is easy, because one compares only two polynomials at a time. Since these two polynomials form a linearly independent set in a two-dimensional space, they form a basis for H by the Basis Theorem.

25. You would have to know that the solution set of the homogeneous system is spanned by two solutions. In this case, the null space of the 18×20 coefficient matrix A is at most two-dimensional. By the Rank Theorem, rank $A = 20 -$ nullity $A \geq 20 - 2 = 18$. Since Col A is a subspace of $\mathbb{R}^{18}$, Col $A = \mathbb{R}^{18}$. Thus $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^{18}$.

26. If $n = 0$, then H and V are both the zero subspace, and $H = V$. If $n > 0$, then a basis for H consists of n linearly independent vectors $\mathbf{u}_1, \dots, \mathbf{u}_n$. These vectors are also linearly independent as elements of V. But since $\dim V = n$, any set of n linearly independent vectors in V must be a basis for V by the Basis Theorem. So $\mathbf{u}_1, \dots, \mathbf{u}_n$ span V, and $H = \text{Span}\{\mathbf{u}_1, \dots, \mathbf{u}_n\} = V$.

27. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the $m \times n$ standard matrix of T.

- **a**. If T is one-to-one, then the columns of A are linearly independent by Theoerm 12 in Section 1.9, so dimNul $A = 0$. By the Rank Theorem, dimCol $A = n - 0 = n$, which is the number of columns of A. As noted in Section 4.2, the range of T is Col A, so the dimension of the range of T is n.
- **b**. If T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$, then the columns of A span $\mathbb{R}^m$ by Theoerm 12 in Section 1.9, so dimCol $A = m$. By the Rank Theorem, dimNul $A = n - m$. As noted in Section 4.2, the kernel of T is Nul A, so the dimension of the kernel of T is $n - m$. Note that $n - m$ must be nonnegative in this case: since A must have a pivot in each row, $n \geq m$.

28. Let $S = \{\mathbf{v}_1, \dots, \mathbf{v}_p\}$. If S were linearly independent and not a basis for V, then S would not span V. In this case, there would be a vector $\mathbf{v}_{p+1}$ in V that is not in $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$. Let $S' = \{\mathbf{v}_1, \dots, \mathbf{v}_p, \mathbf{v}_{p+1}\}$. Then S' is linearly independent since none of the vectors in S' is a linear combination of vectors that precede it. Since S' has more elements than S, this would contradict the maximality of S. Hence S must be a basis for V.

29. If S is a finite spanning set for V, then a subset of S is a basis for V. Denote this subset of S by S'. Since S' is a basis for V, S' must span V. Since S is a minimal spanning set, S' cannot be a proper subset of S. Thus $S' = S$, and S is a basis for V.

30. **a**. Let $\mathbf{y}$ be in Col AB. Then $\mathbf{y} = AB\mathbf{x}$ for some $\mathbf{x}$. But $AB\mathbf{x} = A(B\mathbf{x})$, so $\mathbf{y} = A(B\mathbf{x})$, and $\mathbf{y}$ is in Col A. Thus Col AB is a subspace of Col A, so rank AB = dimCol $AB \leq$ dimCol A = rank A by Theorem 11 in Section 4.5.

- **b**. By the Rank Theorem and part a.: $\text{rank } AB = \text{rank}(AB)^T = \text{rank } B^T A^T \leq \text{rank } B^T = \text{rank } B$

31. By Exercise 30, rank $PA \leq$ rank A, and $\text{rank } A = \text{rank}(P^{-1}P)A = \text{rank } P^{-1}(PA) \leq \text{rank } PA$, so rank PA = rank A.

32. Note that $(AQ)^T = Q^T A^T$. Since Q^T is invertible, we can use Exercise 31 to conclude that $\text{rank}(AQ)^T = \text{rank } Q^T A^T = \text{rank } A^T$. Since the ranks of a matrix and its transpose are equal (by the Rank Theorem), rank $AQ = \text{rank } A$.

33. The equation $AB = 0$ shows that each column of B is in Nul A. Since Nul A is a subspace of $\mathbb{R}^n$, all linear combinations of the columns of B are in Nul A. That is, Col B is a subspace of Nul A. By Theorem 12 in Section 4.5, rank B = dimCol $B \le$ dimNul A. By this inequality and the Rank Theorem applied to A, $n = \text{rank } A + \text{dimNul } A \ge \text{rank } A + \text{rank } B$

34. Suppose that $\text{rank } A = r_1$ and $\text{rank } B = r_2$. Then there are rank factorizations $A = C_1R_1$ and $B = C_2R_2$ of A and B, where C_1 is $m \times r_1$ with rank r_1, C_2 is $m \times r_2$ with rank r_2, R_1 is $r_1 \times n$ with rank r_1, and R_2 is $r_2 \times n$ with rank r_2. Create an $m \times (r_1 + r_2)$ matrix $C = [C_1 \quad C_2]$ and an $(r_1 + r_2) \times n$ matrix R by stacking R_1 over R_2. Then $A + B = C_1R_1 + C_2R_2 = [C_1 \quad C_2]\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = CR$.

Since the matrix CR is a product, its rank cannot exceed the rank of either of its factors by Exercise 30. Since C has $r_1 + r_2$ columns, the rank of C cannot exceed $r_1 + r_2$. Likewise R has $r_1 + r_2$ rows, so the rank of R cannot exceed $r_1 + r_2$. Thus the rank of $A + B$ cannot exceed $r_1 + r_2 = \text{rank } A + \text{rank } B$, or rank $(A + B) \le$ rank A + rank B.

35. Let A be an $m \times n$ matrix with rank r.

(a) Let A_1 consist of the r pivot columns of A. The columns of A_1 are linearly independent, so A_1 is an $m \times r$ matrix with rank r.

(b) By the Rank Theorem applied to A_1, the dimension of $\text{Row}A_1$ is r, so A_1 has r linearly independent rows. Let A_2 consist of the r linearly independent rows of A_1. Then A_2 is an $r \times r$ matrix with linearly independent rows. By the Invertible Matrix Theorem, A_2 is invertible.

36. Let A be a 4×4 matrix and B be a 4×2 matrix, and let $\mathbf{u}_0, \dots, \mathbf{u}_3$ be a sequence of input vectors in $\mathbb{R}^2$.

a. Use the equation $\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k$ for $k = 0, \dots, 4$, $k = 0, \dots, 4$, with $\mathbf{x}_0 = \mathbf{0}$.

$$\mathbf{x}_1 = A\mathbf{x}_0 + B\mathbf{u}_0 = B\mathbf{u}_0$$

$$\mathbf{x}_2 = A\mathbf{x}_1 + B\mathbf{u}_1 = AB\mathbf{u}_0 + B\mathbf{u}_1$$

$$\mathbf{x}_3 = A\mathbf{x}_2 + B\mathbf{u}_2 = A(AB\mathbf{u}_0 + B\mathbf{u}_1) + B\mathbf{u}_2 = A^2B\mathbf{u}_0 + AB\mathbf{u}_1 + B\mathbf{u}_2$$

$$\mathbf{x}_4 = A\mathbf{x}_3 + B\mathbf{u}_3 = A(A^2B\mathbf{u}_0 + AB\mathbf{u}_1 + B\mathbf{u}_2) + B\mathbf{u}_3 = A^3B\mathbf{u}_0 + A^2B\mathbf{u}_1 + AB\mathbf{u}_2 + B\mathbf{u}_3$$

$$= \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}\begin{bmatrix} \mathbf{u}_3 \\ \mathbf{u}_2 \\ \mathbf{u}_1 \\ \mathbf{u}_0 \end{bmatrix} = M\,\mathbf{u}$$

Note that M has 4 rows because B does, and that M has 8 columns because B and each of the matrices A^kB have 2 columns. The vector $\mathbf{u}$ in the final equation is in $\mathbb{R}^8$, because each $\mathbf{u}_k$ is in $\mathbb{R}^2$.

b. If (A, B) is controllable, then the controlability matrix has rank 4, with a pivot in each row, and the columns of M span $\mathbb{R}^4$. Therefore, for any vector **v** in $\mathbb{R}^4$, there is a vector **u** in $\mathbb{R}^8$ such that **v** = M**u**. However, from part a. we know that $\mathbf{x}_4 = M\,\mathbf{u}$ when **u** is partitioned into a control sequence $\mathbf{u}_0,\ldots,\mathbf{u}_3$. This particular control sequence makes $\mathbf{x}_4 = \mathbf{v}$.

37. To determine if the matrix pair (A, B) is controllable, we compute the rank of the matrix $\begin{bmatrix} B & AB & A^2B \end{bmatrix}$. To find the rank, we row reduce:

$$\begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -.9 & .81 \\ 1 & .5 & .25 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The rank of the matrix is 3, and the pair (A, B) is controllable.

38. To determine if the matrix pair (A, B) is controllable, we compute the rank of the matrix $\begin{bmatrix} B & AB & A^2B \end{bmatrix}$. To find the rank, we note that : $\begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & .5 & .19 \\ 1 & .7 & .45 \\ 0 & 0 & 0 \end{bmatrix}$.

The rank of the matrix must be less than 3, and the pair (A, B) is not controllable.

39. To determine if the matrix pair (A, B) is controllable, we compute the rank of the matrix $\begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$. To find the rank, we row reduce:

$$\begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 1.6 \\ 0 & -1 & 1.6 & -.96 \\ -1 & 1.6 & -.96 & -.024 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1.6 \\ 0 & 0 & 1 & -1.6 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The rank of the matrix is 3, and the pair (A, B) is not controllable.

40. To determine if the matrix pair (A, B) is controllable, we compute the rank of the matrix $\begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix}$. To find the rank, we row reduce:

$$\begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & .5 \\ 0 & -1 & .5 & 11.45 \\ -1 & .5 & 11.45 & -10.275 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The rank of the matrix is 4, and the pair (A, B) is controllable.