5 Eigenvalues and Eigenvectors

5.1 – Eigenvalues and Eigenvectors

Notes: Exercises 1–6 reinforce the definitions of eigenvalues and eigenvectors. The subsection on eigenvectors and difference equations, along with Exercises 41 and 42, refers to the chapter introductory example and anticipates discussions of dynamical systems in Sections 5.2 and 5.6.

1. The number 2 is an eigenvalue of A if and only if the equation $A\mathbf{x} = 2\mathbf{x}$ has a nontrivial solution. This equation is equivalent to $(A-2I)\mathbf{x} = \mathbf{0}$. Compute $A - 2I = \begin{bmatrix} 3 & 2 \\ 3 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$. The columns of A are obviously linearly dependent, so $(A-2I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution, and so 2 is an eigenvalue of A.

2. The number -2 is an eigenvalue of A if and only if the equation $A\mathbf{x} = -2\mathbf{x}$ has a nontrivial solution. This equation is equivalent to $(A+2I)\mathbf{x} = \mathbf{0}$. Compute $A + 2I = \begin{bmatrix} 7 & 3 \\ 3 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$.

The columns of A are obviously linearly dependent, so $(A+2I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution, and so -2 is an eigenvalue of A.

3. Is $A\mathbf{x}$ a multiple of $\mathbf{x}$? Compute $\begin{bmatrix} -3 & 1 \\ -3 & 8 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 29 \end{bmatrix} \neq \lambda \begin{bmatrix} 1 \\ 4 \end{bmatrix}$. So $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ is *not* an eigenvector of A.

4. Is $A\mathbf{x}$ a multiple of $\mathbf{x}$? Compute $\begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. The eigenvalue is 2.

5. Is $A\mathbf{x}$ a multiple of $\mathbf{x}$? Compute $\begin{bmatrix} 3 & 7 & 9 \\ -4 & -5 & 1 \\ 2 & 4 & 4 \end{bmatrix}\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. So $\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$ is an eigenvector of A for the eigenvalue 0.

6. Is $A\mathbf{x}$ a multiple of $\mathbf{x}$? Compute $\begin{bmatrix} 2 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \\ -3 \end{bmatrix} = (-3)\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ So $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ is an eigenvector of A for the eigenvalue -3.

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7. To determine if 4 is an eigenvalue of A, decide if the matrix $A-4I$ is invertible.

$A-4I=\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}-\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}=\begin{bmatrix} -1 & 0 & -1 \\ 2 & -1 & 1 \\ -3 & 4 & 1 \end{bmatrix}$. Invertibility can be checked in several ways, but since an eigenvector is needed in the event that one exists, the best strategy is to row reduce the augmented matrix for $(A-4I)\mathbf{x}=\mathbf{0}$:

$\begin{bmatrix} -1 & 0 & -1 & 0 \\ 2 & -1 & 1 & 0 \\ -3 & 4 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 4 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The equation $(A-4I)\mathbf{x}=\mathbf{0}$ has a nontrivial solution, so 4 is an eigenvalue. Any nonzero solution of $(A-4I)\mathbf{x}=\mathbf{0}$ is a corresponding eigenvector. The entries in a solution satisfy $x_1+x_3=0$ and $-x_2-x_3=0,$ with x_3 free. The general solution is *not* requested, so to save time, simply take any nonzero value for x_3 to produce an eigenvector. If $x_3=1,$ then $\mathbf{x}=(-1,-1,1).$

Note: The answer in the text is $(1,1,-1),$ written in this form to make the students wonder whether the more common answer given above is also correct. This may initiate a class discussion of what answers are "correct."

8. To determine if 3 is an eigenvalue of A, decide if the matrix $A-3I$ is invertible.

$A-3I=\begin{bmatrix} 1 & 2 & 2 \\ 3 & -2 & 1 \\ 0 & 1 & 1 \end{bmatrix}-\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}=\begin{bmatrix} -2 & 2 & 2 \\ 3 & -5 & 1 \\ 0 & 1 & -2 \end{bmatrix}$. Row reducing the augmented matrix

$[(\mathrm{A}-3I) \quad \mathbf{0}]$ yields: $\begin{bmatrix} -2 & 2 & 2 & 0 \\ 3 & -5 & 1 & 0 \\ 0 & 1 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -2 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The equation $(A-3I)\mathbf{x}=\mathbf{0}$ has a nontrivial solution, so 3 is an eigenvalue. Any nonzero solution of $(A-3I)\mathbf{x}=\mathbf{0}$ is a corresponding eigenvector. The entries in a solution satisfy $x_1-3x_3=0$ and $x_2-2x_3=0,$ with x_3 free. The general solution is *not* requested, so to save time, simply take any nonzero value for x_3 to produce an eigenvector. If $x_3=1,$ then $\mathbf{x}=\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}.$

9. For $\lambda=1$: $A-1I=\begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix}-\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} 4 & 0 \\ 2 & 0 \end{bmatrix}$. The augmented matrix for $(A-I)\mathbf{x}=\mathbf{0}$ is $\begin{bmatrix} 4 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$. Thus $x_1=0$ and x_2 is free. The general solution of $(A-I)\mathbf{x}=\mathbf{0}$ is $x_2\mathbf{e}_2,$ where $\mathbf{e}_2=\begin{bmatrix} 0 \\ 1 \end{bmatrix},$ and so $\mathbf{e}_2$ is a basis for the eigenspace corresponding to the eigenvalue 1.

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For $\lambda = 5$: $A - 5I = \begin{bmatrix} 5 & 0 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & -4 \end{bmatrix}$. The equation $(A-5I)\mathbf{x} = \mathbf{0}$ leads to $2x_1 - 4x_2 = 0$, so that $x_1 = 2x_2$ and x_2 is free. The general solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. So $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is a basis for the eigenspace.

10. For $\lambda = 4$: $A - 4I = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 6 & -9 \\ 4 & -6 \end{bmatrix}$. The augmented matrix for $(A-4I)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} 6 & -9 & 0 \\ 4 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -9/6 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus $x_1 = (3/2)x_2$ and x_2 is free. The general solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (3/2)x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$. A basis for the eigenspace corresponding to 4 is $\begin{bmatrix} 3/2 \\ 1 \end{bmatrix}$. Another choice is $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

11. $A - 10I = \begin{bmatrix} 4 & -2 \\ -3 & 9 \end{bmatrix} - \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} -6 & -2 \\ -3 & -1 \end{bmatrix}$. The augmented matrix for $(A-10I)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} -6 & -2 & 0 \\ -3 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus $x_1 = (-1/3)x_2$ and x_2 is free. The general solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -(1/3)x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$. A basis for the eigenspace corresponding to 10 is $\begin{bmatrix} -1/3 \\ 1 \end{bmatrix}$. Another choice is $\begin{bmatrix} -1 \\ 3 \end{bmatrix}$.

12. For $\lambda = -2$: $A - (-2)I = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix}$. The augmented matrix for $(A-(-2)I)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} 3 & 4 & 0 \\ 3 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 4/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus $x_1 = (-4/3)x_2$ and x_2 is free. A basis for the eigenspace corresponding to 1 is $\begin{bmatrix} -4/3 \\ 1 \end{bmatrix}$. Another choice is $\begin{bmatrix} -4 \\ 3 \end{bmatrix}$.

For $\lambda = 5$: $A - 5I = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} -4 & 4 \\ 3 & -3 \end{bmatrix}$. The augmented matrix for $(A-5I)\mathbf{x} = \mathbf{0}$ is $\begin{bmatrix} -4 & 4 & 0 \\ 3 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus $x_1 = x_2$ and x_2 is free. The general solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. A basis for the eigenspace is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

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13. For $\lambda = 1$: $A - 1I = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}$. The equations for $(A - I)\mathbf{x} = \mathbf{0}$ are easy to solve: $\left\{ \begin{aligned} 3x_1 + x_3 &= 0 \\ -2x_1 \quad &= 0 \end{aligned} \right\}$. Row operations hardly seem necessary. Obviously x_1 is zero, and hence x_3 is also zero. There are three-variables, so x_2 is free. The general solution of $(A - I)\mathbf{x} = \mathbf{0}$ is $x_2\mathbf{e}_2$, where $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and so $\mathbf{e}_2$ provides a basis for the eigenspace.

For λ = 2: $A - 2I = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{bmatrix}$.

$[(A - 2I) \;\; \mathbf{0}] = \begin{bmatrix} 2 & 0 & 1 & 0 \\ -2 & -1 & 0 & 0 \\ -2 & 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. So $x_1 = -(1/2)x_3, x_2 = x_3$, with x_3 free. The general solution of $(A - 2I)\mathbf{x} = \mathbf{0}$ is $x_3 \begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix}$. A nice basis vector for the eigenspace is $\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$.

For λ = 3: $A - 3I = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{bmatrix}$.

$[(A - 3I) \;\; \mathbf{0}] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -2 & -2 & 0 & 0 \\ -2 & 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. So $x_1 = -x_3, x_2 = x_3$, with x_3 free. A basis vector for the eigenspace is $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$.

14. For $\lambda = -4$: $A - (-4I) = A + 4I = \begin{bmatrix} 3 & -1 & 3 \\ -1 & 3 & 3 \\ 6 & 6 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 7 & -1 & 3 \\ -1 & 7 & 3 \\ 6 & 6 & 6 \end{bmatrix}$. The augmented matrix for $[A - (-4)I]\mathbf{x} = \mathbf{0}$, or $(A + 4I)\mathbf{x} = \mathbf{0}$, is

$[(A + 4I) \;\; \mathbf{0}] = \begin{bmatrix} 7 & -1 & 3 & 0 \\ -1 & 7 & 3 & 0 \\ 6 & 6 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 8 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Thus

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$x_1 = -(1/2)x_3$, $x_2 = -(1/2)x_3$, with x_3 free. The general solution of $(A+4I)\mathbf{x} = \mathbf{0}$ is $x_3\begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$. A basis for the eigenspace corresponding to -4 is $\begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$; another is $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$.

15. For $\lambda = 3$: $[(A-3I)\ \ \mathbf{0}] = \begin{bmatrix} 1 & 2 & 3 & 0 \\ -1 & -2 & -3 & 0 \\ 2 & 4 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Thus $x_1 + 2x_2 + 3x_3 = 0,$ with x_2 and x_3 free. The general solution of $(A-3I)\mathbf{x} = \mathbf{0},$ is $\mathbf{x} = \begin{bmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$. A basis for the eigenspace is: $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Note: For simplicity, the text answer omits the set brackets. I permit my students to list a basis without the set brackets. Some instructors may prefer to include brackets.

16. For

$$\lambda = 4: \quad A - 4I = \begin{bmatrix} 3 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$[(A-4I)\ \ \mathbf{0}] = \begin{bmatrix} -1 & 0 & 2 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. So $x_1 = 2x_3, x_2 = 3x_3$, with x_3 and x_4 free variables. The general solution of $(A-4I)\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ 3x_3 \\ x_3 \\ x_4 \end{bmatrix} = x_3\begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. A basis for the eigenspace is: $\left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$.

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Note: I urge my students always to include the extra column of zeros when solving a homogeneous system. Exercise 16 provides a situation in which *failing* to add the column is likely to create problems for a student, because the matrix $A-4I$ itself has a column of zeros.

17. The eigenvalues of $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 5 \\ 0 & 0 & -1 \end{bmatrix}$ are 0, 2, and -1, on the main diagonal, by Theorem 1.

18. The eigenvalues of $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -3 \end{bmatrix}$ are 4, 0, and -3, on the main diagonal, by Theorem 1.

19. The matrix $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ is not invertible because its columns are linearly dependent. So the number 0 is an eigenvalue of the matrix. See the discussion following Example 5.

20. The matrix $A=\begin{bmatrix} 5 & -5 & 5 \\ 5 & -5 & 5 \\ 5 & -5 & 5 \end{bmatrix}$ is not invertible because its columns are linearly dependent. So the number 0 is an eigenvalue of A. Eigenvectors for the eigenvalue 0 are solutions of $A\mathbf{x}=\mathbf{0}$ and therefore have entries that produce a linear dependence relation among the columns of A. Any nonzero vector (in $\mathbb{R}^3$) whose first and third entries, minus the second, sum to 0, will work. Find any two such vectors that are not multiples; for instance, $(1,1,0)$ and $(0,1,1)$.

21. False. The equation $A\mathbf{x}=\lambda\mathbf{x}$ must have a *nontrivial* solution.

22. False. The vector **x** in $A\mathbf{x}=\lambda\mathbf{x}$ must be *nonzero*.

23. False. See the paragraph after Example 5.

24. True. See the discussion of equation (3).

25. True. See Example 2 and the paragraph preceding it. Also, see the Numerical Note.

26. False. See the warning after Example 3.

27. False. See Example 4 for a two-dimensional eigenspace, which contains two linearly independent eigenvectors corresponding to the same eigenvalue. The statement given is not at all the same as Theorem 2. In fact, it is the *converse* of Theorem 2 (for the case $r=2$).

28. False. Theorem 1 concerns a *triangular* matrix. See Examples 3 and 4 for counterexamples.

29. False. See the remarks prior to Example 4. The eigenvalue does not change when the eigenvector is scaled.

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30. True. See the paragraph following Example 3. The eigenspace of A corresponding to λ is the null space of the matrix $A-\lambda I$.

31. If a 2×2 matrix A were to have three distinct eigenvalues, then by Theorem 2 there would correspond three linearly independent eigenvectors (one for each eigenvalue). This is impossible because the vectors all belong to a two-dimensional vector space, in which any set of three vectors is linearly dependent. See Theorem 8 in Section 1.7. In general, if an $n\times n$ matrix has p distinct eigenvalues, then by Theorem 2 there would be a linearly independent set of p eigenvectors (one for each eigenvalue). Since these vectors belong to an n-dimensional vector space, p cannot exceed n.

32. A simple example of a 2×2 matrix with only one distinct eigenvalue is a triangular matrix with the same number on the diagonal. By experimentation, one finds that if such a matrix is actually a diagonal matrix then the eigenspace is two dimensional, and otherwise the eigenspace is only one dimensional.

Examples: $\begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$ and $\begin{bmatrix} 4 & 5 \\ 0 & 4 \end{bmatrix}$.

33. If λ is an eigenvalue of A, then there is a nonzero vector $\mathbf{x}$ such that $A\mathbf{x}=\lambda\mathbf{x}$. Since A is invertible, $A^{-1}A\mathbf{x}=A^{-1}(\lambda\mathbf{x})$, and so $\mathbf{x}=\lambda(A^{-1}\mathbf{x})$. Since $\mathbf{x}\neq\mathbf{0}$ (and since A is invertible), λ cannot be zero. Then $\lambda^{-1}\mathbf{x}=A^{-1}\mathbf{x}$, which shows that λ^{-1} is an eigenvalue of A^{-1}.

Note: The *Study Guide* points out here that the relation between the eigenvalues of A and A^{-1} is important in the so-called *inverse power method* for estimating an eigenvalue of a matrix. See Section 5.8.

34. Suppose that A^2 is the zero matrix. If $A\mathbf{x}=\lambda\mathbf{x}$ for some $\mathbf{x}\neq\mathbf{0}$, then $A^2\mathbf{x}=A(A\mathbf{x})=A(\lambda\mathbf{x})=\lambda A\mathbf{x}=\lambda^2\mathbf{x}$. Since $\mathbf{x}$ is nonzero, λ must be zero. Thus each eigenvalue of A is zero.

35. Use the *Hint* in the text to write, for any $\lambda, (A-\lambda I)^T=A^T-(\lambda I)^T=A^T-\lambda I$. Since $(A-\lambda I)^T$ is invertible if and only if $A-\lambda I$ is invertible (by Theorem 6(c) in Section 2.2), it follows that $A^T-\lambda I$ is *not* invertible if and only if $A-\lambda I$ is *not* invertible. That is, λ is an eigenvalue of A^T if and only if λ is an eigenvalue of A.

Note: If you discuss Exercise 35, you might ask students on a test to show that A and A^T have the same characteristic polynomial (discussed in Section 5.2). Since det $A=\det A^T$, for any square matrix A, $\det(A-\lambda I)=\det(A-\lambda I)^T=\det(A^T-(\lambda I)^T)=\det(A^T-\lambda I)$.

36. If A is lower triangular, then A^T is upper triangular and has the same diagonal entries as A. Hence, by the part of Theorem 1 already proved in the text, these diagonal entries are eigenvalues of A^T. By Exercise 35, they are also eigenvalues of A.

37. Let $\mathbf{v}$ be the vector in $\mathbb{R}^n$ whose entries are all ones. Then $A\mathbf{v}=s\mathbf{v}$.

38. Suppose the column sums of an $n\times n$ matrix A all equal the same number s. By Exercise 37 applied to A^T in place of A, the number s is an eigenvalue of A^T. By Exercise 35, s is an eigenvalue of A.

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39. Suppose T reflects points across (or through) a line that passes through the origin. That line consists of all multiples of some nonzero vector $\mathbf{v}$. The points on this line do not move under the action of A. So $T(\mathbf{v}) = \mathbf{v}$. If A is the standard matrix of T, then $A\mathbf{v} = \mathbf{v}$. Thus $\mathbf{v}$ is an eigenvector of A corresponding to the eigenvalue 1. The eigenspace is Span $\{\mathbf{v}\}$. Another eigenspace is generated by any nonzero vector $\mathbf{u}$ that is perpendicular to the given line. (Perpendicularity in $\mathbf{R}^2$ should be a familiar concept even though orthogonality in $\mathbf{R}^n$ has not been discussed yet.) Each vector $\mathbf{x}$ on the line through $\mathbf{u}$ is transformed into the vector $-\mathbf{x}$. The eigenvalue is -1.

40. (The solution is given in the text.)

41. **a**. Replace k by $k+1$ in the definition of $\mathbf{x}_k$, and obtain $\mathbf{x}_{k+1} = c_1\lambda^{k+1}\mathbf{u} + c_2\mu^{k+1}\mathbf{v}$.

b.
$$\begin{aligned} A\mathbf{x}_k &= A(c_1\lambda^k\mathbf{u} + c_2\mu^k\mathbf{v}) \\ &= c_1\lambda^k A\mathbf{u} + c_2\mu^k A\mathbf{v} \quad \text{by linearity} \\ &= c_1\lambda^k \lambda\mathbf{u} + c_2\mu^k \mu\mathbf{v} \quad \text{since } \mathbf{u} \text{ and } \mathbf{v} \text{ are eigenvectors} \\ &= \mathbf{x}_{k+1} \end{aligned}$$

42. You could try to write $\mathbf{x}_0$ as linear combination of eigenvectors, $\mathbf{v}_1, \dots, \mathbf{v}_p$. If $\lambda_1, \dots, \lambda_p$ are corresponding eigenvalues, and if $\mathbf{x}_0 = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$, then you could *define*

$$\mathbf{x}_k = c_1\lambda_1^k\mathbf{v}_1 + \cdots + c_p\lambda_p^k\mathbf{v}_p$$

In this case, for $k = 0, 1, 2, \dots,$

$$\begin{aligned} A\mathbf{x}_k &= A(c_1\lambda_1^k\mathbf{v}_1 + \cdots + c_p\lambda_p^k\mathbf{v}_p) \\ &= c_1\lambda_1^k A\mathbf{v}_1 + \cdots + c_p\lambda_p^k A\mathbf{v}_p \quad \text{Linearity} \\ &= c_1\lambda_1^{k+1}\mathbf{v}_1 + \cdots + c_p\lambda_p^{k+1}\mathbf{v}_p \quad \text{The } \mathbf{v}_i \text{ are eigenvectors.} \\ &= \mathbf{x}_{k+1} \end{aligned}$$

43. Using the figure in the exercise, plot $T(\mathbf{u})$ as $2\mathbf{u}$, because $\mathbf{u}$ is an eigenvector for the eigenvalue 2 of the standard matrix A. Likewise, plot $T(\mathbf{v})$ as $3\mathbf{v}$, because $\mathbf{v}$ is an eigenvector for the eigenvalue 3. Since T is linear, the image of $\mathbf{w}$ is $T(\mathbf{w}) = T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$.

44. As in Exercise 43, $T(\mathbf{u}) = -\mathbf{u}$ and $T(\mathbf{v}) = 3\mathbf{v}$ because $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors for the eigenvalues -1 and 3, respectively, of the standard matrix A. Since T is linear, the image of $\mathbf{w}$ is $T(\mathbf{w}) = T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$.

Note: The matrix programs supported by this text all have an eigenvalue command. In some cases, such as MATLAB, the command can be structured so it provides eigenvectors as well as a list of the eigenvalues. At this point in the course, students should *not* use the extra power that produces eigenvectors. Students need to be reminded frequently that eigenvectors of A are null vectors of a translate of A. That is why the instructions for Exercises 45–48 tell students to use the method of Example 4.

It is my experience that nearly all students need manual practice finding eigenvectors by the method of Example 4, at least in this section if not also in Sections 5.2 and 5.3. However, computer exercises do create a burden if eigenvectors must be found manually. For this reason, the data files for the text include a special command, `nulbasis` for each matrix program (MATLAB, Maple, etc.). The output of

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`nulbasis (A)` is a matrix whose columns provide a basis for the null space of A, and these columns are identical to the ones a student would find by row reducing the augmented matrix $[A\ \mathbf{0}]$. With nulbasis, student answers will be the same (up to multiples) as those in the text. I encourage my students to use technology to speed up all numerical homework here, not just those labeled as computer exercises,

45. Let A be the given matrix. Use the MATLAB commands `eig` and `nulbasis` (or equivalent commands). The command `ev = eig(A)` computes the three eigenvalues of A and stores them in a vector `ev`. In this exercise, `ev` $=(3,13,13)$. The eigenspace for the eigenvalue 3 is the null space of $A-3I$. Use `nulbasis` to produce a basis for each null space. If the format is set for rational display, the result is `nulbasis(A-ev(1)*eye(3))` $=\begin{bmatrix}5/9\\-2/9\\1\end{bmatrix}$. For simplicity, scale the entries by 9. A basis for the eigenspace for $\lambda=3$: $\begin{bmatrix}5\\-2\\9\end{bmatrix}$

For the next eigenvalue, 13, compute `nulbasis(A-ev(2)*eye(3))` $=\begin{bmatrix}-2&-1\\1&0\\0&1\end{bmatrix}$. Basis for eigenspace for $\lambda=13$: $\left\{\begin{bmatrix}-2\\1\\0\end{bmatrix},\begin{bmatrix}-1\\0\\1\end{bmatrix}\right\}$. There is no need to use `ev(3)` because it is the same as `ev(2)`.

46. `ev = eig(A)` $=(13,-12,-12,13)$. For $\lambda=13$:

`nulbasis (A-ev(1)*eye(4))` $=\begin{bmatrix}-1/2&1/3\\0&-4/3\\1&0\\0&1\end{bmatrix}$. Basis for eigenspace: $\left\{\begin{bmatrix}-1\\0\\2\\0\end{bmatrix},\begin{bmatrix}1\\-4\\0\\3\end{bmatrix}\right\}$.

For $\lambda=-12$: `nulbasis(A-ev(2)*eye(4))` $=\begin{bmatrix}2/7&0\\1&-1\\1&0\\0&1\end{bmatrix}$. Basis: $\left\{\begin{bmatrix}2\\7\\7\\0\end{bmatrix},\begin{bmatrix}0\\-1\\0\\1\end{bmatrix}\right\}$.

47. For $\lambda=5$, basis: $\left\{\begin{bmatrix}2\\-1\\1\\0\\0\end{bmatrix},\begin{bmatrix}-1\\1\\0\\1\\0\end{bmatrix},\begin{bmatrix}2\\0\\0\\0\\1\end{bmatrix}\right\}$. For $\lambda=-2$, basis: $\left\{\begin{bmatrix}-2\\7\\-5\\5\\0\end{bmatrix},\begin{bmatrix}3\\7\\-5\\0\\5\end{bmatrix}\right\}$

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48. `ev = eig(A)` = $(21.68984106239549, -16.68984106239549, 3, 2, 2)$. The first two eigenvalues are the roots of $\lambda^2 - 5\lambda - 362 = 0$.

$$\text{Basis for } \lambda = \text{ev}(1): \begin{bmatrix} -0.33333333333333 \\ 2.39082008853296 \\ 0.33333333333333 \\ 0.58333333333333 \\ 1.00000000000000 \end{bmatrix}, \text{ for } \lambda = \text{ev}(2): \begin{bmatrix} -0.33333333333333 \\ -0.80748675519962 \\ 0.33333333333333 \\ 0.58333333333333 \\ 1.00000000000000 \end{bmatrix}.$$

$$\text{For the eigenvalues 3 and 2, the eigenbases are } \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -.5 \\ .5 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}, \text{ respectively.}$$

Note: Since so many eigenvalues in text problems are small integers, it is easy for students to form a habit of entering a value for λ in `nulbasis (A - λI)` based on a *visual examination* of the eigenvalues produced by `eig(A)` when only a few decimal places for λ are displayed. Exercise 48 may help your students discover the dangers of this approach.

5.2 - The Characteristic Equation

Notes: Exercises 9–14 can be omitted, unless you want your students to have some facility with determinants of 3×3 matrices. In later sections, the text will provide eigenvalues when they are needed for matrices larger than 2×2. If you discussed partitioned matrices in Section 2.4, you might wish to bring in Supplementary Exercises 34–36 in Chapter 5. (Also, see Exercise 16 of Section 2.4.)

Exercises 33 and 35 support the subsection on dynamical systems. The calculations in these exercises and Example 5 prepare for the discussion in Section 5.6 about eigenvector decompositions.

1. $A = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}, A - \lambda I = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & 7 \\ 7 & 2-\lambda \end{bmatrix}$. The characteristic polynomial is $\det(A - \lambda I) = (2-\lambda)^2 - 7^2 = 4 - 4\lambda + \lambda^2 - 49 = \lambda^2 - 4\lambda - 45$. In factored form, the characteristic equation is $(\lambda - 9)(\lambda + 5) = 0$, so the eigenvalues of A are 9 and -5.

2. $A = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}, A - \lambda I = \begin{bmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{bmatrix}$. The characteristic polynomial is $\det(A - \lambda I) = (5-\lambda)(5-\lambda) - 3\cdot(3) = \lambda^2 - 10\lambda + 16$. Since $\lambda^2 - 10\lambda + 16 = (\lambda - 8)(\lambda - 2)$, the eigenvalues of A are 8 and 2.

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3. $A=\begin{bmatrix}3 & -2\\ 1 & -1\end{bmatrix}$, $A-\lambda I=\begin{bmatrix}3-\lambda & -2\\ 1 & -1-\lambda\end{bmatrix}$. The characteristic polynomial is $\det(A-\lambda I)=(3-\lambda)(-1-\lambda)-(-2)(1)=\lambda^2-2\lambda-1$. Use the quadratic formula to solve the characteristic equation and find the eigenvalues: $\lambda=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}=\dfrac{2\pm\sqrt{4+4}}{2}=1\pm\sqrt{2}$.

4. $A=\begin{bmatrix}4 & -3\\ -4 & 2\end{bmatrix}$, $A-\lambda I=\begin{bmatrix}4-\lambda & -3\\ -4 & 2-\lambda\end{bmatrix}$. The characteristic polynomial of A is $\det(A-\lambda I)=(4-\lambda)(2-\lambda)-(-3)(-4)=\lambda^2-6\lambda-4$. Use the quadratic formula to solve the characteristic equation and find the eigenvalues: $\lambda=\dfrac{6\pm\sqrt{36-4(-4)}}{2}=\dfrac{6\pm 2\sqrt{13}}{2}=3\pm\sqrt{13}$

5. $A=\begin{bmatrix}2 & 1\\ -1 & 4\end{bmatrix}$, $A-\lambda I=\begin{bmatrix}2-\lambda & 1\\ -1 & 4-\lambda\end{bmatrix}$. The characteristic polynomial of A is $\det(A-\lambda I)=(2-\lambda)(4-\lambda)-(1)(-1)=\lambda^2-6\lambda+9=(\lambda-3)^2$. Thus, A has only one eigenvalue 3, with multiplicity 2.

6. $A=\begin{bmatrix}1 & -4\\ 4 & 6\end{bmatrix}$, $A-\lambda I=\begin{bmatrix}1-\lambda & -4\\ 4 & 6-\lambda\end{bmatrix}$. The characteristic polynomial is $\det(A-\lambda I)=(1-\lambda)(6-\lambda)-(-4)(4)=\lambda^2-7\lambda+22$. Use the quadratic formula to solve $\det\ (A-\lambda I)=0$: $\lambda=\dfrac{7\pm\sqrt{49-4(22)}}{2}=\dfrac{7\pm\sqrt{-39}}{2}$. These values are complex numbers, not real numbers, so A has no real eigenvalues. There is no nonzero vector $\mathbf{x}$ in $\mathbb{R}^2$ such that $A\mathbf{x}=\lambda\mathbf{x}$, because a real vector $A\mathbf{x}$ cannot equal a complex multiple of $\mathbf{x}$.

7. $A=\begin{bmatrix}5 & 3\\ -4 & 4\end{bmatrix}$, $A-\lambda I=\begin{bmatrix}5-\lambda & 3\\ -4 & 4-\lambda\end{bmatrix}$. The characteristic polynomial is $\det(A-\lambda I)=(5-\lambda)(4-\lambda)-(3)(-4)=\lambda^2-9\lambda+32$. Use the quadratic formula to solve $\det\ (A-\lambda I)=0$: $\lambda=\dfrac{9\pm\sqrt{81-4(32)}}{2}=\dfrac{9\pm\sqrt{-47}}{2}$. These values are complex numbers, not real numbers, so A has no real eigenvalues. There is no nonzero vector $\mathbf{x}$ in $\mathbb{R}^2$ such that $A\mathbf{x}=\lambda\mathbf{x}$, because a real vector $A\mathbf{x}$ cannot equal a complex multiple of $\mathbf{x}$.

8. $A=\begin{bmatrix}7 & -2\\ 2 & 3\end{bmatrix}$, $A-\lambda I=\begin{bmatrix}7-\lambda & -2\\ 2 & 3-\lambda\end{bmatrix}$. The characteristic polynomial is $\det(A-\lambda I)=(7-\lambda)(3-\lambda)-(-2)(2)=\lambda^2-10\lambda+25$. Since $\lambda^2-10\lambda+25=(\lambda-5)^2$, the only eigenvalue is 5, with multiplicity 2.

9. $\det(A-\lambda I)=\det\begin{bmatrix}1-\lambda & 0 & -1\\ 2 & 3-\lambda & -1\\ 0 & 6 & 0-\lambda\end{bmatrix}$. The characteristic polynomial is

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$$\det(A-\lambda I)=(1-\lambda)\det\begin{bmatrix}3-\lambda & -1\\ 6 & -\lambda\end{bmatrix}-(2)\det\begin{bmatrix}0 & -1\\ 6 & -\lambda\end{bmatrix}$$
$$=(1-\lambda)\big((3-\lambda)(-\lambda)+6\big)-2(6)$$
$$=(1-\lambda)\left(\lambda^2-3\lambda+6\right)-12$$
$$=-\lambda^3+4\lambda^2-9\lambda-6$$

(This polynomial has one irrational zero and two imaginary zeros.) Another way to evaluate the determinant is to interchange rows 1 and 2 (which reverses the sign of the determinant) and then make one row replacement:

$$\det\begin{bmatrix}1-\lambda & 0 & -1\\ 2 & 3-\lambda & -1\\ 0 & 6 & 0-\lambda\end{bmatrix}=-\det\begin{bmatrix}2 & 3-\lambda & -1\\ 1-\lambda & 0 & -1\\ 0 & 6 & 0-\lambda\end{bmatrix}$$

$$=-\det\begin{bmatrix}2 & 3-\lambda & -1\\ 0 & 0+(.5\lambda-.5)(3-\lambda) & -1+(.5\lambda-.5)(-1)\\ 0 & 6 & 0-\lambda\end{bmatrix}.$$ Next, expand by cofactors down the first column. The quantity above equals

$$-2\det\begin{bmatrix}(.5\lambda-.5)(3-\lambda) & -.5-.5\lambda\\ 6 & -\lambda\end{bmatrix}=-2[(.5\lambda-.5)(3-\lambda)(-\lambda)-(-.5-.5\lambda)(6)]$$
$$=(1-\lambda)(3-\lambda)(-\lambda)-(1+\lambda)(6)=(\lambda^2-4\lambda+3)(-\lambda)-6-6\lambda=-\lambda^3+4\lambda^2-9\lambda-6$$

10. $\det(A-\lambda I)=\det\begin{bmatrix}0-\lambda & 3 & 1\\ 3 & 0-\lambda & 2\\ 1 & 2 & 0-\lambda\end{bmatrix}$. The characteristic polynomial is

$$\det(A-\lambda I)=(-\lambda)\det\begin{bmatrix}-\lambda & 2\\ 2 & -\lambda\end{bmatrix}-3\det\begin{bmatrix}3 & 1\\ 2 & -\lambda\end{bmatrix}+1\det\begin{bmatrix}3 & 1\\ -\lambda & 2\end{bmatrix}$$
$$=-\lambda\left(\lambda^2-4\right)-3\left(-3\lambda-2\right)+\left(6+\lambda\right)=-\lambda^3+14\lambda+12$$

11. The special arrangements of zeros in A makes a cofactor expansion along the first row highly effective.

$$\det(A-\lambda I)=\det\begin{bmatrix}4-\lambda & 0 & 0\\ 5 & 3-\lambda & 2\\ -2 & 0 & 2-\lambda\end{bmatrix}=(4-\lambda)\det\begin{bmatrix}3-\lambda & 2\\ 0 & 2-\lambda\end{bmatrix}$$
$$=(4-\lambda)(3-\lambda)(2-\lambda)=(4-\lambda)(\lambda^2-5\lambda+6)=-\lambda^3+9\lambda^2-26\lambda+24$$

If only the eigenvalues were required, there would be no need here to write the characteristic polynomial in expanded form.

12. Make a cofactor expansion along the third row:

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$$\det(A-\lambda I)=\det\begin{bmatrix}1-\lambda & 0 & 1\\ -3 & 6-\lambda & 1\\ 0 & 0 & 4-\lambda\end{bmatrix}=(4-\lambda)\cdot\det\begin{bmatrix}1-\lambda & 0\\ -3 & 6-\lambda\end{bmatrix}$$

$$=(4-\lambda)(1-\lambda)(6-\lambda)=-\lambda^3+11\lambda^2-34\lambda+24$$

13. Make a cofactor expansion down the third column:

$$\det(A-\lambda I)=\det\begin{bmatrix}6-\lambda & -2 & 0\\ -2 & 9-\lambda & 0\\ 5 & 8 & 3-\lambda\end{bmatrix}=(3-\lambda)\cdot\det\begin{bmatrix}6-\lambda & -2\\ -2 & 9-\lambda\end{bmatrix}$$

$$=(3-\lambda)[(6-\lambda)(9-\lambda)-(-2)(-2)]=(3-\lambda)(\lambda^2-15\lambda+50)$$

$$=-\lambda^3+18\lambda^2-95\lambda+150 \text{ or } (3-\lambda)(\lambda-5)(\lambda-10)$$

14. Make a cofactor expansion along the second row:

$$\det(A-\lambda I)=\det\begin{bmatrix}3-\lambda & -2 & 3\\ 0 & -1-\lambda & 0\\ 6 & 7 & -4-\lambda\end{bmatrix}=(-1-\lambda)\cdot\det\begin{bmatrix}3-\lambda & 3\\ 6 & -4-\lambda\end{bmatrix}$$

$$=(-1-\lambda)\cdot[(3-\lambda)(-4-\lambda)-3\cdot 6]=(-1-\lambda)(\lambda^2+\lambda-30)$$

$$=-\lambda^3-2\lambda^2+29\lambda+30 \text{ or } (-1-\lambda)(\lambda+6)(\lambda-5)$$

15. Use the fact that the determinant of a triangular matrix is the product of the diagonal entries:

$$\det(A-\lambda I)=\det\begin{bmatrix}4-\lambda & -7 & 0 & 2\\ 0 & 3-\lambda & -4 & 6\\ 0 & 0 & 3-\lambda & -8\\ 0 & 0 & 0 & 1-\lambda\end{bmatrix}=(4-\lambda)(3-\lambda)^2(1-\lambda)$$

The eigenvalues are 4, 3, 3, and 1.

16. The determinant of a triangular matrix is the product of its diagonal entries:

$$\det(A-\lambda I)=\det\begin{bmatrix}5-\lambda & 0 & 0 & 0\\ 8 & -4-\lambda & 0 & 0\\ 0 & 7 & 1-\lambda & 0\\ 1 & -5 & 2 & 1-\lambda\end{bmatrix}=(5-\lambda)(-4-\lambda)(1-\lambda)^2$$

The eigenvalues are 5, 1, 1, and –4.

17. The determinant of a triangular matrix is the product of its diagonal entries:

$$\begin{bmatrix}3-\lambda & 0 & 0 & 0 & 0\\ -5 & 1-\lambda & 0 & 0 & 0\\ 3 & 8 & 0-\lambda & 0 & 0\\ 0 & -7 & 2 & 1-\lambda & 0\\ -4 & 1 & 9 & -2 & 3-\lambda\end{bmatrix}=(3-\lambda)^2(1-\lambda)^2(-\lambda)$$

The eigenvalues are 3, 3, 1, 1, and 0.

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18. Row reduce the augmented matrix for the equation $(A-5I)\mathbf{x}=\mathbf{0}$:

$$\begin{bmatrix} 0 & -2 & 6 & -1 & 0 \\ 0 & -2 & h & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & -2 & 6 & -1 & 0 \\ 0 & 0 & h-6 & 1 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & h-6 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For a two-dimensional eigenspace, the system above needs two free variables. This happens if and only if $h=6$.

19. Since the equation $\det(A-\lambda I)=(\lambda_1-\lambda)(\lambda_2-\lambda)\cdots(\lambda_n-\lambda)$ holds for all λ, set $\lambda=0$ and conclude that $\det A=\lambda_1\lambda_2\cdots\lambda_n$.

20. $\det(A^T-\lambda I)=\det(A^T-\lambda I^T)$

$=\det(A-\lambda I)^T$ Transpose property

$=\det(A-\lambda I)$ Theorem 3(c)

21. False. See IMT.

22. True. The eigenspace is a subspace.

23. False. See Exercise 20.

24. True. See the comment before Theorem 4.

25. True. Recall the definition of eigenvector and eigenvalue.

26. False. See the warning after Theorem 4.

27. False. The number -5 is the eigenvalue.

28. True. See the comments before Example 4.

29. True. This is an easy calculation.

30. False. An *n*-th degree polynomial has at most *n* roots.

31. If $A=QR$, with Q invertible, and if $A_1=RQ$, then write $A_1=Q^{-1}QRQ=Q^{-1}AQ$, which shows that A_1 is similar to A.

32. First, observe that if P is invertible, then Theorem 3(b) shows that
$1=\det I=\det(PP^{-1})=(\det P)(\det P^{-1})$. Use Theorem 3(b) again when $A=PBP^{-1}$,
$\det A=\det(PBP^{-1})=(\det P)(\det B)(\det P^{-1})=(\det B)(\det P)(\det P^{-1})=\det B$.

33. Answers will vary, but should show that the eigenvectors of A are not the same as the eigenvectors of A^T, unless, of course, $A^T=A$.

34. Answers will vary. The product of the eigenvalues of A should equal $\det A$.

35. The characteristic polynomials and the eigenvalues for the various values of a are given in the following table:

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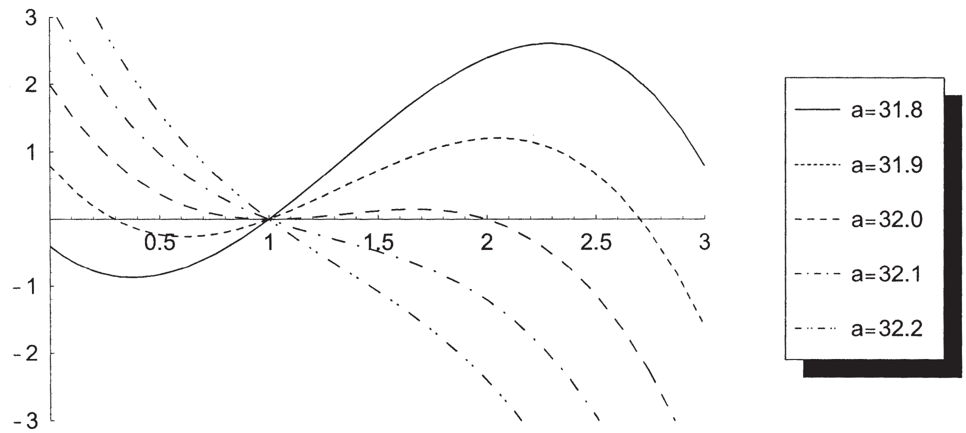
a	Characteristic Polynomial	Eigenvalues
31.8	$-.4-2.6t+4t^2-t^3$	$3.1279, 1, -.1279$
31.9	$.8-3.8t+4t^2-t^3$	2.7042, 1, .2958
32.0	$2-5t+4t^2-t^3$	2, 1, 1
32.1	$3.2-6.2t+4t^2-t^3$	$1.5\pm.9747i, 1$
32.2	$4.4-7.4t+4t^2-t^3$	$1.5\pm1.4663i, 1$

The graphs of the characteristic polynomials are:



Notes: An appendix in Section 5.3 of the *Study Guide* gives an example of factoring a cubic polynomial with integer coefficients, in case you want your students to find integer eigenvalues of simple 3×3 or perhaps 4×4 matrices.

The MATLAB box for Section 5.3 introduces the command `poly (A)`, which lists the coefficients of the characteristic polynomial of the matrix A, and it gives MATLAB code that will produce a graph of the characteristic polynomial. (This is needed for Exercise 35.) The Maple and Mathematica appendices have corresponding information. The appendices for the TI and HP calculators contain only the commands that list the coefficients of the characteristic polynomial.

5.3 - Diagonalization

1. $P=\begin{bmatrix}5 & 7\\ 2 & 3\end{bmatrix}, D=\begin{bmatrix}2 & 0\\ 0 & 1\end{bmatrix}, A=PDP^{-1}$, and $A^4=PD^4P^{-1}$. We compute

$$P^{-1}=\begin{bmatrix}3 & -7\\ -2 & 5\end{bmatrix}, D^4=\begin{bmatrix}16 & 0\\ 0 & 1\end{bmatrix}, \text{ and } A^4=\begin{bmatrix}5 & 7\\ 2 & 3\end{bmatrix}\begin{bmatrix}16 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}3 & -7\\ -2 & 5\end{bmatrix}=\begin{bmatrix}226 & -525\\ 90 & -209\end{bmatrix}$$

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2. $P=\begin{bmatrix}2 & -3\\ -3 & 5\end{bmatrix}, D=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, A=PDP^{-1}$, and $A^4=PD^4P^{-1}$. We compute

$P^{-1}=\begin{bmatrix}5 & 3\\ 3 & 2\end{bmatrix}, D^4=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, and $A^4=\begin{bmatrix}2 & -3\\ -3 & 5\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}5 & 3\\ 3 & 2\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

3. $A^k=PD^kP^{-1}=\begin{bmatrix}1 & 0\\ 3 & 1\end{bmatrix}\begin{bmatrix}a^k & 0\\ 0 & b^k\end{bmatrix}\begin{bmatrix}1 & 0\\ -3 & 1\end{bmatrix}=\begin{bmatrix}a^k & 0\\ 3a^k-3b^k & b^k\end{bmatrix}.$

4.

$$A^k=PD^kP^{-1}=\begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}\begin{bmatrix}2^k & 0\\ 0 & (-2)^k\end{bmatrix}\begin{bmatrix}-1 & 2\\ 1 & -1\end{bmatrix}=\begin{bmatrix}-(2^k)+2(-2)^k & 2(2^k)-2(-2)^k\\ -(2^k)+(-2)^k & 2(2^k)-(-2)^k\end{bmatrix}=2^k\begin{bmatrix}-1+2(-1)^k & 2-2(-1)^k\\ -1+(-1)^k & 2-(-1)^k\end{bmatrix}$$

$$=\begin{cases}2^k\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \text{if } k \text{ is even}\\ 2^k\begin{bmatrix}-3 & 4\\ -2 & 3\end{bmatrix}, \text{if } k \text{ is odd}\end{cases}$$

5. By the Diagonalization Theorem, eigenvectors form the columns of the left factor, and they correspond respectively to the eigenvalues on the diagonal of the middle factor.

$\lambda=5:\begin{bmatrix}1\\ 1\\ 1\end{bmatrix};\lambda=1:\begin{bmatrix}1\\ 0\\ -1\end{bmatrix},\begin{bmatrix}2\\ -1\\ 0\end{bmatrix}$

6. As in Exercise 5, inspection of the factorization gives: $\lambda=3:\begin{bmatrix}2\\ 1\\ 1\end{bmatrix},\begin{bmatrix}-1\\ -1\\ 0\end{bmatrix};\lambda=4:\begin{bmatrix}-2\\ -1\\ 0\end{bmatrix}$.

7. Since A is triangular, its eigenvalues are obviously ± 1.

<u>For $\lambda = 1$</u>: $A-1I=\begin{bmatrix}0 & 0\\ 6 & -2\end{bmatrix}$. The equation $(A-1I)\mathbf{x}=\mathbf{0}$ amounts to $6x_1-2x_2=0$, so $x_1=(1/3)x_2$ with x_2 free. The general solution is $x_2\begin{bmatrix}1/3\\ 1\end{bmatrix}$, and a nice basis vector for the eigenspace is $\mathbf{v}_1=\begin{bmatrix}1\\ 3\end{bmatrix}$.

<u>For $\lambda = -1$</u>: $A+1I=\begin{bmatrix}2 & 0\\ 6 & 0\end{bmatrix}$. The equation $(A+1I)\mathbf{x}=\mathbf{0}$ amounts to $2x_1=0$, so $x_1=0$ with x_2 free. The general solution is $x_2\begin{bmatrix}0\\ 1\end{bmatrix}$, and a basis vector for the eigenspace is $\mathbf{v}_2=\begin{bmatrix}0\\ 1\end{bmatrix}$.

From $\mathbf{v}_1$ and $\mathbf{v}_2$ construct $P=\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{bmatrix}=\begin{bmatrix}1 & 0\\ 3 & 1\end{bmatrix}$. Then set $D=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively.

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8. Since A is triangular, its only eigenvalue is obviously 5.

For $\lambda = 5$: $A-5I=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$. The equation $(A-5I)\mathbf{x}=\mathbf{0}$ amounts to $x_2=0$, so $x_2=0$ with x_1 free. The general solution is $x_1\begin{bmatrix}1\\ 0\end{bmatrix}$. Since we cannot generate an eigenvector basis for $\mathbb{R}^2$, A is not diagonalizable.

9. To find the eigenvalues of A, compute its characteristic polynomial: $\det(A-\lambda I)=\det\begin{bmatrix}3-\lambda & -1\\ 1 & 5-\lambda\end{bmatrix}=(3-\lambda)(5-\lambda)-(-1)(1)=\lambda^2-8\lambda+16=(\lambda-4)^2$.

Thus the only eigenvalue of A is 4.

For $\lambda = 4$: $A-4I=\begin{bmatrix}-1 & -1\\ 1 & 1\end{bmatrix}$. The equation $(A-4I)\mathbf{x}=\mathbf{0}$ amounts to $x_1+x_2=0$, so $x_1=-x_2$ with x_2 free. The general solution is $x_2\begin{bmatrix}-1\\ 1\end{bmatrix}$. Since we cannot generate an eigenvector basis for $\mathbb{R}^2$, A is not diagonalizable.

10. To find the eigenvalues of A, compute its characteristic polynomial:
$\det(A-\lambda I)=\det\begin{bmatrix}2-\lambda & 3\\ 4 & 1-\lambda\end{bmatrix}=(2-\lambda)(1-\lambda)-(3)(4)=\lambda^2-3\lambda-10=(\lambda-5)(\lambda+2)$.

Thus the eigenvalues of A are 5 and -2.

For $\lambda = 5$: $A-5I=\begin{bmatrix}-3 & 3\\ 4 & -4\end{bmatrix}$. The equation $(A-5I)\mathbf{x}=\mathbf{0}$ amounts to $x_1-x_2=0$, so $x_1=x_2$ with x_2 free. The general solution is $x_2\begin{bmatrix}1\\ 1\end{bmatrix}$, and a basis vector for the eigenspace is $\mathbf{v}_1=\begin{bmatrix}1\\ 1\end{bmatrix}$.

For $\lambda = -2$: $A+2I=\begin{bmatrix}4 & 3\\ 4 & 3\end{bmatrix}$. The equation $(A+1I)\mathbf{x}=\mathbf{0}$ amounts to $4x_1+3x_2=0$, so $x_1=(-3/4)x_2$ with x_2 free. The general solution is $x_2\begin{bmatrix}-3/4\\ 1\end{bmatrix}$, and a nice basis vector for the eigenspace is $\mathbf{v}_2=\begin{bmatrix}-3\\ 4\end{bmatrix}$.

From $\mathbf{v}_1$ and $\mathbf{v}_2$ construct $P=\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{bmatrix}=\begin{bmatrix}1 & -3\\ 1 & 4\end{bmatrix}$. Then set $D=\begin{bmatrix}5 & 0\\ 0 & -2\end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively.

11. The eigenvalues of A are given to be 1, 2, and 3.

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For $\lambda = 3$: $A-3I=\begin{bmatrix}-4 & 4 & -2\\ -3 & 1 & 0\\ -3 & 1 & 0\end{bmatrix}$, and row reducing $\begin{bmatrix}A-3I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1 & 0 & -1/4 & 0\\ 0 & 1 & -3/4 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$. The general solution is $x_3\begin{bmatrix}1/4\\ 3/4\\ 1\end{bmatrix}$, and a nice basis vector for the eigenspace is $\mathbf{v}_1=\begin{bmatrix}1\\ 3\\ 4\end{bmatrix}$.

For $\lambda = 2$: $A-2I=\begin{bmatrix}-3 & 4 & -2\\ -3 & 2 & 0\\ -3 & 1 & 1\end{bmatrix}$, and row reducing $\begin{bmatrix}A-2I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1 & 0 & -2/3 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$. The general solution is $x_3\begin{bmatrix}2/3\\ 1\\ 1\end{bmatrix}$, and a nice basis vector for the eigenspace is $\mathbf{v}_2=\begin{bmatrix}2\\ 3\\ 3\end{bmatrix}$.

For $\lambda = 1$: $A-I=\begin{bmatrix}-2 & 4 & -2\\ -3 & 3 & 0\\ -3 & 1 & 2\end{bmatrix}$, and row reducing $\begin{bmatrix}A-1I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$. The general solution is $x_3\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$, and a basis vector for the eigenspace is $\mathbf{v}_3=\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$.

From $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_3$ construct $P=\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix}=\begin{bmatrix}1 & 2 & 1\\ 3 & 3 & 1\\ 4 & 3 & 1\end{bmatrix}$. Then set $D=\begin{bmatrix}3 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1,\mathbf{v}_2$ and $\mathbf{v}_3$ respectively.

12. The eigenvalues of A are given to be 1 and 4.

For $\lambda = 1$: $A-I=\begin{bmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{bmatrix}$, and row reducing $\begin{bmatrix}A-I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1 & 0 & -1 & 0\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$. The general solution is $x_3\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$, and a basis vector for the eigenspace is $\mathbf{v}_1=\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}$.

For $\lambda = 4$: $A-4I=\begin{bmatrix}-1 & -1 & -1\\ -1 & -1 & -1\\ -1 & -1 & -1\end{bmatrix}$, and row reducing $\begin{bmatrix}A-4I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1 & 1 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$. The general solution is $x_2\begin{bmatrix}-1\\ 1\\ 0\end{bmatrix}+x_3\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}$, and a basis for the eigenspace is $\{\mathbf{v}_2,\mathbf{v}_3\}=\left\{\begin{bmatrix}-1\\ 1\\ 0\end{bmatrix},\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}\right\}$.

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From $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ construct $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Then set $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ respectively.

13. The eigenvalues of A are given to be 5 and 1.

For $\lambda = 5$: $A - 5I = \begin{bmatrix} -3 & 2 & -1 \\ 1 & -2 & -1 \\ -1 & -2 & -3 \end{bmatrix}$, and row reducing $\begin{bmatrix} A - 5I & \mathbf{0} \end{bmatrix}$ yields $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$, and a basis for the eigenspace is $\mathbf{v}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$.

For $\lambda = 1$: $A - 1I = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 2 & -1 \\ -1 & -2 & 1 \end{bmatrix}$, and row reducing $\begin{bmatrix} A - I & \mathbf{0} \end{bmatrix}$ yields $\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and a basis for the eigenspace is $\{\mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$.

From $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ construct $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} -1 & -2 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Then set $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ respectively.

14. The eigenvalues of A are given to be 3 and 4.

For $\lambda = 3$: $A - 3I = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$, and row reducing $\begin{bmatrix} A - 3I & \mathbf{0} \end{bmatrix}$ yields $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$, and a basis for the eigenspace is $\{\mathbf{v}_1, \mathbf{v}_2\} = \left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

For $\lambda = 4$: $A - 4I = \begin{bmatrix} 0 & 0 & 2 \\ 2 & -1 & 4 \\ 0 & 0 & -1 \end{bmatrix}$, and row reducing $\begin{bmatrix} A - 4I & \mathbf{0} \end{bmatrix}$ yields $\begin{bmatrix} 1 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}$, and a nice basis vector for the eigenspace is $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

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From $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ construct $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}$. Then set $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ respectively.

15. The eigenvalues of A are given to be 3 and 1.

For $\lambda = 3$: $A - 3I = \begin{bmatrix} 4 & 4 & 16 \\ 2 & 2 & 8 \\ -2 & -2 & -8 \end{bmatrix}$, and row reducing $\begin{bmatrix} A-3I & \mathbf{0} \end{bmatrix}$ yields $\begin{bmatrix} 1 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$, and a basis for the eigenspace is $\{\mathbf{v}_1, \mathbf{v}_2\} = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} \right\}$

For $\lambda = 1$: $A - I = \begin{bmatrix} 6 & 4 & 16 \\ 2 & 4 & 8 \\ -2 & -2 & -6 \end{bmatrix}$, and row reducing $\begin{bmatrix} A-I & \mathbf{0} \end{bmatrix}$ yields $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$, and a basis for the eigenspace is $\mathbf{v}_3 = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$.

From $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ construct $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} -1 & -4 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$. Then set $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ respectively.

16. The eigenvalues of A are given to be 2 and 1.

For $\lambda = 2$: $A - 2I = \begin{bmatrix} -2 & -4 & -6 \\ -1 & -2 & -3 \\ 1 & 2 & 3 \end{bmatrix}$, and row reducing $\begin{bmatrix} A-2I & \mathbf{0} \end{bmatrix}$ yields $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$, and a basis for the eigenspace is $\{\mathbf{v}_1, \mathbf{v}_2\} = \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

For $\lambda = 1$: $A - I = \begin{bmatrix} -1 & -4 & -6 \\ -1 & -1 & -3 \\ 1 & 2 & 4 \end{bmatrix}$, and row reducing $\begin{bmatrix} A-I & \mathbf{0} \end{bmatrix}$ yields $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$, and a basis for the eigenspace is $\mathbf{v}_3 = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$.

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From $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ construct $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} -2 & -3 & -2 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$. Then set $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ respectively.

17. Since A is triangular, its eigenvalues are obviously 4 and 5.

For $\lambda = 4$: $A - 4I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and row reducing $\begin{bmatrix} A - 4I & \mathbf{0} \end{bmatrix}$ yields $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and a basis for the eigenspace is $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Since $\lambda = 5$ must have only a one-dimensional eigenspace, we can find at most 2 linearly independent eigenvectors for A, so A is not diagonalizable.

18. An eigenvalue of A is given to be 5; an eigenvector $\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$ is also given. To find the eigenvalue corresponding to $\mathbf{v}_1$, compute $A\mathbf{v}_1 = \begin{bmatrix} -7 & -16 & 4 \\ 6 & 13 & -2 \\ 12 & 16 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ -6 \end{bmatrix} = -3\mathbf{v}_1$. Thus the eigenvalue in question is -3.

For $\lambda = 5$: $A - 5I = \begin{bmatrix} -12 & -16 & 4 \\ 6 & 8 & -2 \\ 12 & 16 & -4 \end{bmatrix}$, and row reducing $\begin{bmatrix} A - 5I & \mathbf{0} \end{bmatrix}$ yields

$\begin{bmatrix} 1 & 4/3 & -1/3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_2 \begin{bmatrix} -4/3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1/3 \\ 0 \\ 1 \end{bmatrix}$, and a nice basis for the eigenspace is $\{\mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{bmatrix} -4 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \right\}$.

From $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ construct $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} = \begin{bmatrix} -2 & -4 & 1 \\ 1 & 3 & 0 \\ 2 & 0 & 3 \end{bmatrix}$. Then set $D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ respectively. Note that this answer differs from the text. There, $P = \begin{bmatrix} \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_1 \end{bmatrix}$ and the entries in D are rearranged to match the new order of the eigenvectors. According to the Diagonalization Theorem, both answers are correct.

19. Since A is triangular, its eigenvalues are obviously 2, 3, and 5.

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<u>For λ = 2</u>: $A-2I=\begin{bmatrix}3 & -3 & 0 & 9\\ 0 & 1 & 1 & -2\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$, and row reducing $\begin{bmatrix}A-2I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}1 & 0 & 1 & 1 & 0\\ 0 & 1 & 1 & -2 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$.

The general solution is $x_3\begin{bmatrix}-1\\ -1\\ 1\\ 0\end{bmatrix}+x_4\begin{bmatrix}-1\\ 2\\ 0\\ 1\end{bmatrix}$, and a nice basis for the eigenspace is

$\{\mathbf{v}_1,\mathbf{v}_2\}=\left\{\begin{bmatrix}-1\\ -1\\ 1\\ 0\end{bmatrix},\begin{bmatrix}-1\\ 2\\ 0\\ 1\end{bmatrix}\right\}$.

<u>For λ = 3</u>: $A-3I=\begin{bmatrix}2 & -3 & 0 & 9\\ 0 & 0 & 1 & -2\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{bmatrix}$, and row reducing $\begin{bmatrix}A-3I & \mathbf{0}\end{bmatrix}$ yields

$\begin{bmatrix}1 & -3/2 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$. The general solution is $x_2\begin{bmatrix}3/2\\ 1\\ 0\\ 0\end{bmatrix}$, and a nice basis for the eigenspace is

$\mathbf{v}_3=\begin{bmatrix}3\\ 2\\ 0\\ 0\end{bmatrix}$.

<u>For λ = 5</u>: $A-5I=\begin{bmatrix}0 & -3 & 0 & 9\\ 0 & -2 & 1 & -2\\ 0 & 0 & -3 & 0\\ 0 & 0 & 0 & -3\end{bmatrix}$, and row reducing $\begin{bmatrix}A-5I & \mathbf{0}\end{bmatrix}$ yields $\begin{bmatrix}0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$.

The general solution is $x_1\begin{bmatrix}1\\ 0\\ 0\\ 0\end{bmatrix}$, and a basis for the eigenspace is $\mathbf{v}_4=\begin{bmatrix}1\\ 0\\ 0\\ 0\end{bmatrix}$.

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From $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{v}_4$ construct $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 3 & 1 \\ -1 & 2 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$. Then set

$D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$, where the eigenvalues in D correspond to $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{v}_4$ respectively. Note that this answer differs from the text. There, $P = \begin{bmatrix} \mathbf{v}_4 & \mathbf{v}_3 & \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix}$ and the entries in D are rearranged to match the new order of the eigenvectors. According to the Diagonalization Theorem, both answers are correct.

20. Since A is triangular, its eigenvalues are obviously 2.

<u>For $\lambda = 2$</u>: $A - 2I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, and row reducing $\begin{bmatrix} A - 2I & \mathbf{0} \end{bmatrix}$ yields $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. The general solution is $x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, and a basis for the eigenspace is

$\{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. Since this matrix only has one eigenvalue, it is not diagonalizable.

21. False. The symbol D does not automatically denote a diagonal matrix.

22. True. See the remark after the statement of the Diagonalization Theorem.

23. False. The 3×3 matrix in Example 4 has 3 eigenvalues, counting multiplicities, but it is not diagonalizable.

24. False. Invertibility depends on 0 not being an eigenvalue. (See the Invertible Matrix Theorem.) A diagonalizable matrix may or may not have 0 as an eigenvalue. See Examples 3 and 5 for both possibilities.

25. False. The n eigenvectors must be linearly independent. See the Diagonalization Theorem.

26. False. The matrix in Example 3 is diagonalizable, but it has only 2 distinct eigenvalues. (The statement given is the *converse* of Theorem 6.)

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27. True. This follows from $AP = PD$ and formulas (1) and (2) in the proof of the Diagonalization Theorem.

28. False. See Example 4. The matrix there is invertible because 0 is not an eigenvalue, but the matrix is not diagonalizable.

29. A is diagonalizable because you know that five linearly independent eigenvectors exist: three in the three-dimensional eigenspace and two in the two-dimensional eigenspace. Theorem 7 guarantees that the set of all five eigenvectors is linearly independent.

30. No, by Theorem 7(b). Here is an explanation that does not appeal to Theorem 7: Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be eigenvectors that span the two one-dimensional eigenspaces. If **v** is any other eigenvector, then it belongs to one of the eigenspaces and hence is a multiple of either $\mathbf{v}_1$ or $\mathbf{v}_2$. So there cannot exist three linearly independent eigenvectors. By the Diagonalization Theorem, A cannot be diagonalizable.

31. Let $\{\mathbf{v}_1\}$ be a basis for the one-dimensional eigenspace, let $\mathbf{v}_2$ and $\mathbf{v}_3$ form a basis for the two-dimensional eigenspace, and let $\mathbf{v}_4$ be any eigenvector in the remaining eigenspace. By Theorem 7, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly independent. Since A is 4×4, the Diagonalization Theorem shows that A is diagonalizable.

32. Yes, if the third eigenspace is only one-dimensional. In this case, the sum of the dimensions of the eigenspaces will be six, whereas the matrix is 7×7. See Theorem 7(b). An argument similar to that for Exercise 30 can also be given.

33. If A is diagonalizable, then $A = PDP^{-1}$ for some invertible P and diagonal D. Since A is invertible, 0 is not an eigenvalue of A. So the diagonal entries in D (which are eigenvalues of A) are not zero, and D is invertible. By the theorem on the inverse of a product, $A^{-1} = (PDP^{-1})^{-1} = (P^{-1})^{-1}D^{-1}P^{-1} = PD^{-1}P^{-1}$. Since D^{-1} is obviously diagonal, A^{-1} is diagonalizable.

34. If A has n linearly independent eigenvectors, then by the Diagonalization Theorem, $A = PDP^{-1}$ for some invertible P and diagonal D. Using properties of transposes, $A^T = (PDP^{-1})^T = (P^{-1})^T D^T P^T = (P^T)^{-1}DP^T = QDQ^{-1}$, where $Q = (P^T)^{-1}$. Thus A^T is diagonalizable. By the Diagonalization Theorem, the columns of Q are n linearly independent eigenvectors of A^T.

35. The diagonal entries in D_1 are reversed from those in D. So interchange the (eigenvector) columns of P to make them correspond properly to the eigenvalues in D_1. In this case, $P_1 = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$ and $D_1 = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$. Although the first column of P must be an eigenvector corresponding to the eigenvalue 3, there is nothing to prevent us from selecting some multiple of $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$, say $\begin{bmatrix} -3 \\ 6 \end{bmatrix}$, and

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letting $P_2 = \begin{bmatrix} -3 & 1 \\ 6 & -1 \end{bmatrix}$. We now have three different factorizations or “diagonalizations” of A:

$A = PDP^{-1} = P_1 D_1 P_1^{-1} = P_2 D_1 P_2^{-1}$

36. A nonzero multiple of an eigenvector is another eigenvector. To produce P_2, simply multiply one or both columns of P by a nonzero scalar unequal to 1.

37. For a 2×2 matrix A to be invertible, its eigenvalues must be nonzero. A first attempt at a construction might be something such as $\begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}$, whose eigenvalues are 2 and 4. Unfortunately, a 2×2 matrix with two distinct eigenvalues is diagonalizable (Theorem 6). So, adjust the construction to $\begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$, which works. In fact, any matrix of the form $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ has the desired properties when a and b are nonzero. The eigenspace for the eigenvalue a is one-dimensional, as a simple calculation shows, and there is no other eigenvalue to produce a second eigenvector.

38. Any 2×2 matrix with two distinct eigenvalues is diagonalizable, by Theorem 6. If one of those eigenvalues is zero, then the matrix will not be invertible. Any matrix of the form $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ has the desired properties when a and b are nonzero. The number a must be nonzero to make the matrix diagonalizable; b must be nonzero to make the matrix not diagonal. Other solutions are $\begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix}$ and $\begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix}$.

39. $A = \begin{bmatrix} -6 & 4 & 0 & 9 \\ -3 & 0 & 1 & 6 \\ -1 & -2 & 1 & 0 \\ -4 & 4 & 0 & 7 \end{bmatrix}$, `ev = eig(A)=(5,1,-2,-2)`.

`nulbasis(A-ev(1)*eye(4))` $= \begin{bmatrix} 1.0000 \\ 0.5000 \\ -0.5000 \\ 1.0000 \end{bmatrix}$, A basis for the eigenspace of $\lambda = 5$ is $\begin{bmatrix} 2 \\ 1 \\ -1 \\ 2 \end{bmatrix}$.

`nulbasis(A-ev(2)*eye(4))` $= \begin{bmatrix} 1.0000 \\ -0.5000 \\ -3.5000 \\ 1.0000 \end{bmatrix}$, A basis for the eigenspace of $\lambda = 1$ is $\begin{bmatrix} 2 \\ -1 \\ -7 \\ 2 \end{bmatrix}$.

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`nulbasis(A-ev(3)*eye(4))` $= \begin{bmatrix} 1.0000 \\ 1.0000 \\ 1.0000 \\ 0 \end{bmatrix}, \begin{bmatrix} 1.5000 \\ -0.7500 \\ 0 \\ 1.0000 \end{bmatrix}$, A basis for the eigenspace of

$\lambda = -2$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 0 \\ 4 \end{bmatrix}$.

Thus we construct $P = \begin{bmatrix} 2 & 2 & 1 & 6 \\ 1 & -1 & 1 & -3 \\ -1 & -7 & 1 & 0 \\ 2 & 2 & 0 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$.

40. $A = \begin{bmatrix} 0 & 13 & 8 & 4 \\ 4 & 9 & 8 & 4 \\ 8 & 6 & 12 & 8 \\ 0 & 5 & 0 & -4 \end{bmatrix}$, `ev = eig(A)=(-4,24,1,-4).`

`nulbasis(A-ev(1)*eye(4))` $= \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. A basis for the eigenspace of

$\lambda = -4$ is $\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

`nulbasis(A-ev(2)*eye(4))` $= \begin{bmatrix} 5.6000 \\ 5.6000 \\ 7.2000 \\ 1.0000 \end{bmatrix}$. A basis for the eigenspace of $\lambda = 24$ is $\begin{bmatrix} 28 \\ 28 \\ 36 \\ 5 \end{bmatrix}$.

`nulbasis(A-ev(3)*eye(4))` $= \begin{bmatrix} 1.0000 \\ 1.0000 \\ -2.0000 \\ 1.0000 \end{bmatrix}$. A basis for the eigenspace of $\lambda = 1$ is $\begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}$.

Thus we construct $P = \begin{bmatrix} -2 & -1 & 28 & 1 \\ 0 & 0 & 28 & 1 \\ 1 & 0 & 36 & -2 \\ 0 & 1 & 5 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 24 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

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41. $A=\begin{bmatrix} 11 & -6 & 4 & -10 & -4 \\ -3 & 5 & -2 & 4 & 1 \\ -8 & 12 & -3 & 12 & 4 \\ 1 & 6 & -2 & 3 & -1 \\ 8 & -18 & 8 & -14 & -1 \end{bmatrix}$, `ev = eig(A)=(5,1,3,5,1).`

`nulbasis(A-ev(1)*eye(5))` $= \begin{bmatrix} 2.0000 \\ -0.3333 \\ -1.0000 \\ 1.0000 \\ 0 \end{bmatrix}, \begin{bmatrix} 1.0000 \\ -0.3333 \\ -1.0000 \\ 0 \\ 1.0000 \end{bmatrix}$. A basis for the eigenspace of

$\lambda=5$ is $\begin{bmatrix} 6 \\ -1 \\ -3 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -3 \\ 0 \\ 3 \end{bmatrix}$.

`nulbasis(A-ev(2)*eye(5))` $= \begin{bmatrix} 0.8000 \\ -0.6000 \\ -0.4000 \\ 1.0000 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.6000 \\ -0.2000 \\ -0.8000 \\ 0 \\ 1.0000 \end{bmatrix}$. A basis for the eigenspace of

$\lambda=1$ is $\begin{bmatrix} 4 \\ -3 \\ -2 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -4 \\ 0 \\ 5 \end{bmatrix}$.

`nulbasis(A-ev(3)*eye(5))` $= \begin{bmatrix} 0.5000 \\ -0.2500 \\ -1.0000 \\ -0.2500 \\ 1.0000 \end{bmatrix}$. A basis for the eigenspace of $\lambda=3$ is $\begin{bmatrix} 2 \\ -1 \\ -4 \\ -1 \\ 4 \end{bmatrix}$.

Thus we construct $P=\begin{bmatrix} 6 & 3 & 4 & 3 & 2 \\ -1 & -1 & -3 & -1 & -1 \\ -3 & -3 & -2 & -4 & -4 \\ 3 & 0 & 5 & 0 & -1 \\ 0 & 3 & 0 & 5 & 4 \end{bmatrix}$ and $D=\begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$.

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42. $A=\begin{bmatrix} 4 & 4 & 2 & 3 & -2 \\ 0 & 1 & -2 & -2 & 2 \\ 6 & 12 & 11 & 2 & -4 \\ 9 & 20 & 10 & 10 & -6 \\ 15 & 28 & 14 & 5 & -3 \end{bmatrix}$, `ev = eig(A) = (3,5,7,5,3).`

`nulbasis(A-ev(1)*eye(5))` $=\begin{bmatrix} 2.0000 \\ -1.5000 \\ 0.5000 \\ 1.0000 \\ 0 \end{bmatrix}, \begin{bmatrix} -1.0000 \\ 0.5000 \\ 0.5000 \\ 0 \\ 1.0000 \end{bmatrix}$. A basis for the eigenspace of

$\lambda=3$ is $\begin{bmatrix} 4 \\ -3 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$.

`nulbasis(A-ev(2)*eye(5))` $=\begin{bmatrix} 0 \\ -0.5000 \\ 1.0000 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1.0000 \\ 1.0000 \\ 0 \\ -1.0000 \\ 1.0000 \end{bmatrix}$. A basis for the eigenspace of

$\lambda=5$ is $\begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$.

`nulbasis(A-ev(3)*eye(5))` $=\begin{bmatrix} 0.3333 \\ 0.0000 \\ 0.0000 \\ 1.0000 \\ 1.0000 \end{bmatrix}$. A basis for the eigenspace of $\lambda=7$ is $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \\ 3 \end{bmatrix}$.

Thus we construct $P=\begin{bmatrix} 4 & -2 & 0 & -1 & 1 \\ -3 & 1 & -1 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 \\ 2 & 0 & 0 & -1 & 3 \\ 0 & 2 & 0 & 1 & 3 \end{bmatrix}$ and $D=\begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 7 \end{bmatrix}$.

Notes: For your use, here is another matrix with five distinct real eigenvalues. To four decimal places, they are 11.0654, 9.8785, 3.8238, −3.7332, and −6.0345.

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$$\begin{bmatrix} 6 & -8 & 5 & -3 & 0 \\ -7 & 3 & -5 & 3 & 0 \\ -3 & -7 & 5 & -3 & 5 \\ 0 & -4 & 1 & -7 & 5 \\ -5 & -3 & -2 & 0 & 8 \end{bmatrix}.$$

The MATLAB box in the *Study Guide* encourages students to use `eig (A)` and `nulbasis` to practice the diagonalization procedure in this section. It also remarks that in later work, a student may automate the process, using the command `[P D]= eig (A)`. You may wish to permit students to use the full power of `eig` in some problems in Sections 5.5 and 5.7.

5.4 - Eigenvalues and Linear Transformations

1. $\begin{bmatrix} 3 & -1 & 0 \\ -5 & 6 & 4 \\ 0 & 0 & 0 \end{bmatrix}$

2. $\begin{bmatrix} 2 & -4 \\ -3 & 5 \end{bmatrix}$

3. $T(1) = 3 + 5t; T(t) = -2t + 4t^2; T(t^2) = t^2; [T]_\beta = \begin{bmatrix} 3 & 0 & 0 \\ 5 & -2 & 0 \\ 0 & 4 & 1 \end{bmatrix}$

4. **a**. Notice $T(\mathbf{p}+\mathbf{q}) = (\mathbf{p}+\mathbf{q})(0) - (\mathbf{p}+\mathbf{q})(1)t + (\mathbf{p}+\mathbf{q})(2)t^2 = \mathbf{p}(0) - \mathbf{p}(1)t + \mathbf{p}(2)t^2 + \mathbf{q}(0) - \mathbf{q}(1)t + \mathbf{q}(2)t^2$ $=T(\mathbf{p}) + T(\mathbf{q})$ and $T(c\mathbf{p}) = (c\mathbf{p})(0) - (c\mathbf{p})(1)t + (c\mathbf{p})(2)t^2 = c(\mathbf{p}(0) - \mathbf{p}(1)t + \mathbf{p}(2)t^2) = cT(\mathbf{p})$, hence T is a linear transformation.

b. $T(\mathbf{p}) = -2 + t = \mathbf{p}$, so $\mathbf{p}$ is an eigenvector with eigenvalue 1.

c. $[T]_\beta = \begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & -1 \\ 1 & 2 & 4 \end{bmatrix}$

5. $\begin{bmatrix} 0 & -6 & 1 \\ 0 & 5 & -1 \\ 1 & -2 & 7 \end{bmatrix}\begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix} = \begin{bmatrix} 24 \\ -20 \\ 11 \end{bmatrix}$, so $T(3\mathbf{b}_1 - 4\mathbf{b}_2) = 24\mathbf{b}_1 - 20\mathbf{b}_2 + 11\mathbf{b}_3$.

6. $\begin{bmatrix} 0 & -6 & 1 \\ 0 & 5 & -1 \\ 1 & -2 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ -9 \\ 32 \end{bmatrix}$, so $T(2\mathbf{b}_1 - \mathbf{b}_2 + 4\mathbf{b}_3) = 10\mathbf{b}_1 - 9\mathbf{b}_2 + 32\mathbf{b}_3$.

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7. If $P = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$, then the $\mathcal{B}$-matrix is

$$P^{-1}AP = \frac{1}{5}\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$$

8. If $P = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix}$, then the $\mathcal{B}$-matrix is

$$P^{-1}AP = \frac{1}{5}\begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

9. Start by diagonalizing A. The characteristic polynomial is $\lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3)$, so the eigenvalues of A are 1 and 3.

<u>For $\lambda = 1$:</u> $A - I = \begin{bmatrix} -1 & 1 \\ -3 & 3 \end{bmatrix}$. The equation $(A - I)\mathbf{x} = \mathbf{0}$ amounts to $-x_1 + x_2 = 0$, so $x_1 = x_2$ with x_2 free. A basis vector for the eigenspace is thus $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

<u>For $\lambda = 3$:</u> $A - 3I = \begin{bmatrix} -3 & 1 \\ -3 & 1 \end{bmatrix}$. The equation $(A - 3I)\mathbf{x} = \mathbf{0}$ amounts to $-3x_1 + x_2 = 0$, so $x_1 = (1/3)x_2$ with x_2 free. A nice basis vector for the eigenspace is thus $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

From $\mathbf{v}_1$ and $\mathbf{v}_2$ we may construct $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$ which diagonalizes A. By Theorem 8, the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ has the property that the $\mathcal{B}$-matrix of the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is a diagonal matrix.

10. Start by diagonalizing A. The characteristic polynomial is $\lambda^2 - 6\lambda - 16 = (\lambda - 8)(\lambda + 2)$, so the eigenvalues of A are 8 and -2.

<u>For $\lambda = 8$:</u> $A - 8I = \begin{bmatrix} -3 & -3 \\ -7 & -7 \end{bmatrix}$. The equation $(A - 8I)\mathbf{x} = \mathbf{0}$ amounts to $x_1 + x_2 = 0$, so $x_1 = -x_2$ with x_2 free. A basis vector for the eigenspace is thus $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

<u>For $\lambda = -2$:</u> $A + 2I = \begin{bmatrix} 7 & -3 \\ -7 & 3 \end{bmatrix}$. The equation $(A + 2I)\mathbf{x} = \mathbf{0}$ amounts to $7x_1 - 3x_2 = 0$, so $x_1 = (3/7)x_2$ with x_2 free. A nice basis vector for the eigenspace is thus $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$.

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From $\mathbf{v}_1$ and $\mathbf{v}_2$ we may construct $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1 & 7 \end{bmatrix}$ which diagonalizes A. By Theorem 8, the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ has the property that the $\mathcal{B}$-matrix of the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is a diagonal matrix.

11. Start by diagonalizing A. The characteristic polynomial is $\lambda^2 - 7\lambda + 10 = (\lambda - 5)(\lambda - 2)$, so the eigenvalues of A are 5 and 2.

For $\lambda = 5$: $A - 5I = \begin{bmatrix} -1 & -2 \\ -1 & -2 \end{bmatrix}$. The equation $(A - 5I)\mathbf{x} = \mathbf{0}$ amounts to $x_1 + 2x_2 = 0$, so $x_1 = -2x_2$ with x_2 free. A basis vector for the eigenspace is thus $\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

For $\lambda = 2$: $A - 2I = \begin{bmatrix} 2 & -2 \\ -1 & 1 \end{bmatrix}$. The equation $(A - 2I)\mathbf{x} = \mathbf{0}$ amounts to $x_1 - x_2 = 0$, so $x_1 = x_2$ with x_2 free. A basis vector for the eigenspace is thus $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

From $\mathbf{v}_1$ and $\mathbf{v}_2$ we may construct $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$ which diagonalizes A. By Theorem 8, the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ has the property that the $\mathcal{B}$-matrix of the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is a diagonal matrix.

12. Start by diagonalizing A. The characteristic polynomial is $\lambda^2 - 5\lambda = \lambda(\lambda - 5)$, so the eigenvalues of A are 5 and 0.

For $\lambda = 5$: $A - 5I = \begin{bmatrix} -3 & -6 \\ -1 & -2 \end{bmatrix}$. The equation $(A - 5I)\mathbf{x} = \mathbf{0}$ amounts to $x_1 + 2x_2 = 0$, so $x_1 = -2x_2$ with x_2 free. A basis vector for the eigenspace is thus $\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

For $\lambda = 0$: $A - 0I = \begin{bmatrix} 2 & -6 \\ -1 & 3 \end{bmatrix}$. The equation $(A - 0I)\mathbf{x} = \mathbf{0}$ amounts to $x_1 - 3x_2 = 0$, so $x_1 = 3x_2$ with x_2 free. A basis vector for the eigenspace is thus $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

From $\mathbf{v}_1$ and $\mathbf{v}_2$ we may construct $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 1 & 1 \end{bmatrix}$ which diagonalizes A. By Theorem 8, the basis $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2\}$ has the property that the $\mathcal{B}$-matrix of the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is a diagonal matrix.

13. **a**. We compute that $A\mathbf{b}_1 = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2\mathbf{b}_1$, so $\mathbf{b}_1$ is an eigenvector of A corresponding to the eigenvalue 2. The characteristic polynomial of A is $\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$, so 2 is the only

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eigenvalue for A. Now $A-2I=\begin{bmatrix}-1 & 1\\ -1 & 1\end{bmatrix}$, which implies that the eigenspace corresponding to the eigenvalue 2 is one-dimensional. Thus the matrix A is not diagonalizable.

b. Following Example 5, if $P=\begin{bmatrix}\mathbf{b}_1 & \mathbf{b}_2\end{bmatrix}$, then the $\mathcal{B}$-matrix for T is

$$P^{-1}AP=\begin{bmatrix}-4 & 5\\ 1 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & 3\end{bmatrix}\begin{bmatrix}1 & 5\\ 1 & 4\end{bmatrix}=\begin{bmatrix}2 & -1\\ 0 & 2\end{bmatrix}.$$

14. If there is a basis $\mathcal{B}$ such that $[T]_{\mathcal{B}}$ is diagonal, then A is similar to a diagonal matrix, by the second paragraph following Example 4. For this to happen, A would have three linearly independent eigenvectors. However, this is not necessarily the case, because A has only two distinct eigenvalues.

15. **a**. $T(\mathbf{p}) = 3 + 3t + 3t^2 = 3\,\mathbf{p}$, thus $\mathbf{p}$ is an eigenvector with eigenvalue 3.

b. $T(\mathbf{p}) = -1 - t - t^2$, which is not a multiple of $\mathbf{p}$, so it is not an eigenvector.

16. **a**. $T(\mathbf{p}) = 1 - 4t - 4t^2 + t^3$, which is not a multiple of $\mathbf{p}$, so it is not an eigenvector.

b. $T(\mathbf{p}) = -2t - 2t^2 = -2\mathbf{p}$, thus $\mathbf{p}$ is an eigenvector with eigenvalue -2.

17. True. See Theorem 4 from Section 5.2.

18. False. See Example 4.

19. False. See Example 1.

20. True. See the definition prior to Example 1.

21. If A is similar to B, then there exists an invertible matrix P such that $P^{-1}AP = B$. Thus B is invertible because it is the product of invertible matrices. By a theorem about inverses of products, $B^{-1}=P^{-1}A^{-1}(P^{-1})^{-1}=P^{-1}A^{-1}P$, which shows that A^{-1} is similar to B^{-1}.

22. If $A=PBP^{-1}$, then $A^2=(PBP^{-1})(PBP^{-1})=PB(P^{-1}P)BP^{-1}=PB\cdot I\cdot BP^{-1}=PB^2P^{-1}$. So A^2 is similar to B^2.

23. By hypothesis, there exist invertible P and Q such that $P^{-1}BP=A$ and $Q^{-1}CQ=A$. Then $P^{-1}BP=Q^{-1}CQ$. Left-multiply by Q and right-multiply by Q^{-1} to obtain $QP^{-1}BPQ^{-1}=QQ^{-1}CQQ^{-1}$. So $C=QP^{-1}BPQ^{-1}=(PQ^{-1})^{-1}B(PQ^{-1})$, which shows that B is similar to C.

24. If A is diagonalizable, then $A=PDP^{-1}$ for some invertible matrix P and diagonal matrix D. Also, if B is similar to A, then $B=QAQ^{-1}$ for some Q. Then $B=Q(PDP^{-1})Q^{-1}=(QP)D(P^{-1}Q^{-1})=(QP)D(QP)^{-1}$. So B is diagonalizable.

25. If $A\mathbf{x}=\lambda\mathbf{x}, \mathbf{x}\neq 0$, then $P^{-1}A\mathbf{x}=\lambda P^{-1}\mathbf{x}$. If $B=P^{-1}AP$, then

$$B(P^{-1}\mathbf{x})=P^{-1}AP(P^{-1}\mathbf{x})=P^{-1}A\mathbf{x}=\lambda P^{-1}\mathbf{x} \qquad (*)$$

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by the first calculation. Note that $P^{-1}\mathbf{x} \neq 0$, because $\mathbf{x} \neq 0$ and P^{-1} is invertible. Hence (*) shows that $P^{-1}\mathbf{x}$ is an eigenvector of B corresponding to λ. (Of course, λ is an eigenvalue of both A and B because the matrices are similar, by Theorem 4 in Section 5.2.)

26. If $A = PBP^{-1}$, then $\text{rank } A = \text{rank } P(BP^{-1}) = \text{rank } BP^{-1}$, by Supplementary Exercise 13 in Chapter 4. Also, $\text{rank } BP^{-1} = \text{rank } B$, by Supplementary Exercise 32 in Chapter 4, since P^{-1} is invertible. Thus $\text{rank } A = \text{rank } B$.

27. If $A = PBP^{-1}$, then

$$\begin{aligned} \text{tr}(A) &= \text{tr}((PB)P^{-1}) = \text{tr}(P^{-1}(PB)) \qquad \text{By the trace property} \\ &= \text{tr}(P^{-1}PB) = \text{tr}(IB) = \text{tr}(B) \end{aligned}$$

If B is diagonal, then the diagonal entries of B must be the eigenvalues of A, by the Diagonalization Theorem (Theorem 5 in Section 5.3). So $\text{tr } A = \text{tr } B = \{\text{sum of the eigenvalues of } A\}$.

28. If $A = PDP^{-1}$ for some P, then the general trace property from Exercise 27 shows that $\text{tr } A = \text{tr }[(PD)P^{-1}] = \text{tr }[P^{-1}PD] = \text{tr } D$. (Or, one can use the result of Exercise 27 that since A is similar to D, $\text{tr } A = \text{tr } D$.) Since the eigenvalues of A are on the main diagonal of D, tr D is the sum of the eigenvalues of A.

29. $S(\chi) = \chi$, so χ is an eigenvector of S with eigenvalue 1.

30. $S(\alpha) = -\alpha$, so α is an eigenvector of S with eigenvalue -1.

31. $M_2(\alpha) = \{0\} = 0\alpha$, so α is an eigenvector of M_2 with eigenvalue 0.

32. $M_2(\chi) = \chi$, so χ is an eigenvector of M_2 with eigenvalue 1.

33. If P is the matrix whose columns come from $\mathcal{B}$ then the $\mathcal{B}$-matrix of the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is $D = P^{-1}AP$. From the data in the text,

$$A = \begin{bmatrix} -14 & 4 & -14 \\ -33 & 9 & -31 \\ 11 & -4 & 11 \end{bmatrix}, P = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \end{bmatrix} = \begin{bmatrix} -1 & -1 & -1 \\ -2 & -1 & -2 \\ 1 & 1 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} 2 & -1 & 1 \\ -2 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} -14 & 4 & -14 \\ -33 & 9 & -31 \\ 11 & -4 & 11 \end{bmatrix} \begin{bmatrix} -1 & -1 & -1 \\ -2 & -1 & -2 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 3 & -6 \\ 0 & 1 & 3 \\ 0 & 0 & -3 \end{bmatrix}$$

34. If P is the matrix whose columns come from $\mathcal{B}$, then the $\mathcal{B}$-matrix of the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is $D = P^{-1}AP$. From the data in the text,

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$$A=\begin{bmatrix}-7&-48&-16\\1&14&6\\-3&-45&-19\end{bmatrix}, P=\begin{bmatrix}\mathbf{b}_1&\mathbf{b}_2&\mathbf{b}_3\end{bmatrix}=\begin{bmatrix}-3&-2&3\\1&1&-1\\-3&-3&0\end{bmatrix},$$

$$D=\begin{bmatrix}-1&-3&-1/3\\1&3&0\\0&-1&-1/3\end{bmatrix}\begin{bmatrix}-7&-48&-16\\1&14&6\\-3&-45&-19\end{bmatrix}\begin{bmatrix}-3&-2&3\\1&1&-1\\-3&-3&0\end{bmatrix}=\begin{bmatrix}-7&-2&-6\\0&-4&-6\\0&0&-1\end{bmatrix}$$

35. $A=\begin{bmatrix}15&-66&-44&-33\\0&13&21&-15\\1&-15&-21&12\\2&-18&-22&8\end{bmatrix}$, `ev = eig(A) = (2, 4, 4, 5).`

`nulbasis(A-ev(1)*eye(4))` $=\begin{bmatrix}0.0000\\-1.5000\\1.5000\\1.0000\end{bmatrix}$. A basis for the eigenspace of $\lambda=2$ is

$\mathbf{b}_1=\begin{bmatrix}0\\-3\\3\\2\end{bmatrix}$.

`nulbasis(A-ev(2)*eye(4))` $=\begin{bmatrix}-10.0000\\-2.3333\\1.0000\\0\end{bmatrix},\begin{bmatrix}13.0000\\1.6667\\0\\1.0000\end{bmatrix}$. A basis for the eigenspace of

$\underline{\lambda=4}$ is $\{\mathbf{b}_2,\mathbf{b}_3\}=\left\{\begin{bmatrix}-30\\-7\\3\\0\end{bmatrix},\begin{bmatrix}39\\5\\0\\3\end{bmatrix}\right\}$.

`nulbasis(A-ev(4)*eye(4))` $=\begin{bmatrix}2.7500\\-0.7500\\1.0000\\1.0000\end{bmatrix}$. A basis for the eigenspace of $\lambda=5$ is

$\mathbf{b}_4=\begin{bmatrix}11\\-3\\4\\4\end{bmatrix}$.

The basis $\mathcal{B}=\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4\}$ is a basis for $\mathbb{R}^4$ with the property that $[T]_{\mathcal{B}}$ is diagonal.

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Note: The *Study Guide* comments on Exercise 27 and tells students that the trace of *any* square matrix A equals the sum of the eigenvalues of A, counted according to multiplicities. This provides a quick check on the accuracy of an eigenvalue calculation. You could also refer students to the property of the determinant described in Exercise 19 of Section 5.2.

5.5 - Complex Eigenvalues

1. $A=\begin{bmatrix}1 & -2\\ 1 & 3\end{bmatrix}$, $A-\lambda I=\begin{bmatrix}1-\lambda & -2\\ 1 & 3-\lambda\end{bmatrix}$. $\det(A-\lambda I)=(1-\lambda)(3-\lambda)-(-2)=\lambda^2-4\lambda+5$. Use the quadratic formula to find the eigenvalues: $\lambda=\frac{4\pm\sqrt{16-20}}{2}=2\pm i$. Example 2 gives a shortcut for finding one eigenvector, and Example 5 shows how to write the other eigenvector with no effort.

For $\lambda = 2 + i$: $A-(2+i)I=\begin{bmatrix}-1-i & -2\\ 1 & 1-i\end{bmatrix}$. The equation $(A-\lambda I)\mathbf{x}=\mathbf{0}$ gives

$$\begin{aligned}(-1-i)x_1-2x_2&=0\\ x_1+(1-i)x_2&=0\end{aligned}$$

As in Example 2, the two equations are equivalent—each determines the same relation between x_1 and x_2. So use the second equation to obtain $x_1=-(1-i)x_2$, with x_2 free. The general solution is $x_2\begin{bmatrix}-1+i\\ 1\end{bmatrix}$, and the vector $\mathbf{v}_1=\begin{bmatrix}-1+i\\ 1\end{bmatrix}$ provides a basis for the eigenspace.

For $\lambda = 2 - i$: Let $\mathbf{v}_2=\overline{\mathbf{v}}_1=\begin{bmatrix}-1-i\\ 1\end{bmatrix}$. The remark prior to Example 5 shows that $\mathbf{v}_2$ is automatically an eigenvector for $\overline{2+i}$. In fact, calculations similar to those above would show that $\{\mathbf{v}_2\}$ is a basis for the eigenspace. (In general, for a real matrix A, it can be shown that the set of complex conjugates of the vectors in a basis of the eigenspace for λ is a basis of the eigenspace for $\overline{\lambda}$.)

2. $A=\begin{bmatrix}-1 & -1\\ 5 & -5\end{bmatrix}$. The characteristic polynomial is $\lambda^2+6\lambda+10$, so the eigenvalues of A are $\lambda=\frac{-6\pm\sqrt{36-40}}{2}=-3\pm i$.

For $\lambda = -3 + i$: $A-(-3+i)I=\begin{bmatrix}2-i & -1\\ 5 & -2-i\end{bmatrix}$. The equation $(A-(-3+i)I)\mathbf{x}=\mathbf{0}$ amounts to $5x_1+(-2-i)x_2=0$, so $x_1=\frac{1}{5}(2+i)x_2$ with x_2 free. A basis vector for the eigenspace is thus $\mathbf{v}_1=\begin{bmatrix}2+i\\ 5\end{bmatrix}$.

For $\lambda = -3 - i$: A basis vector for the eigenspace is $\mathbf{v}_2=\overline{\mathbf{v}}_1=\begin{bmatrix}2-i\\ 5\end{bmatrix}$.

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3. $A=\begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix}$. The characteristic polynomial is $\lambda^2-4\lambda+13$, so the eigenvalues of A are $\lambda=\dfrac{4\pm\sqrt{-36}}{2}=2\pm 3i.$

For $\lambda = 2+3i$: $A-(2+3i)I=\begin{bmatrix} -1-3i & 5 \\ -2 & 1-3i \end{bmatrix}$. The equation $(A-(2+3i)I)\mathbf{x}=\mathbf{0}$ amounts to $-2x_1+(1-3i)x_2=0$, so $x_1=\dfrac{1-3i}{2}x_2$ with x_2 free. A nice basis vector for the eigenspace is thus $\mathbf{v}_1=\begin{bmatrix} 1-3i \\ 2 \end{bmatrix}$.

For $\lambda = 2-3i$: A basis vector for the eigenspace is $\mathbf{v}_2=\overline{\mathbf{v}}_1=\begin{bmatrix} 1+3i \\ 2 \end{bmatrix}$.

4. $A=\begin{bmatrix} -3 & -1 \\ 2 & -5 \end{bmatrix}$. The characteristic polynomial is $\lambda^2+8\lambda+17$, so the eigenvalues of A are $\lambda=\dfrac{-8\pm\sqrt{-4}}{2}=-4\pm i.$

For $\lambda = -4+i$: $A-(-4+i)I=\begin{bmatrix} 1-i & -1 \\ 2 & -1-i \end{bmatrix}$. The equation $(A-(-4+i)I)\mathbf{x}=\mathbf{0}$ amounts to $2x_1+(-1-i)x_2=0$, so $x_1=\dfrac{1}{2}(1+i)x_2$ with x_2 free. A basis vector for the eigenspace is thus $\mathbf{v}_1=\begin{bmatrix} 1+i \\ 2 \end{bmatrix}$.

For $\lambda = -4-i$: A basis vector for the eigenspace is $\mathbf{v}_2=\overline{\mathbf{v}}_1=\begin{bmatrix} 1-i \\ 2 \end{bmatrix}$.

5. $A=\begin{bmatrix} 0 & 1 \\ -8 & 4 \end{bmatrix}$. The characteristic polynomial is $\lambda^2-4\lambda+8$, so the eigenvalues of A are $\lambda=\dfrac{4\pm\sqrt{-16}}{2}=2\pm 2i.$

For $\lambda = 2+2i$: $A-(2+2i)I=\begin{bmatrix} -2-2i & 1 \\ -8 & 2-2i \end{bmatrix}$. The equation $(A-(2+2i)I)\mathbf{x}=\mathbf{0}$ amounts to $(-2-2i)x_1+x_2=0$, so $x_2=(2+2i)x_1$ with x_1 free. A basis vector for the eigenspace is thus $\mathbf{v}_1=\begin{bmatrix} 1 \\ 2+2i \end{bmatrix}$.

For $\lambda = 2-2i$: A basis vector for the eigenspace is $\mathbf{v}_2=\overline{\mathbf{v}}_1=\begin{bmatrix} 1 \\ 2-2i \end{bmatrix}$.

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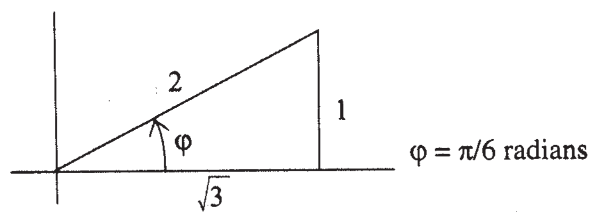
6. $A=\begin{bmatrix}4 & 3\\ -3 & 4\end{bmatrix}$. The characteristic polynomial is $\lambda^2-8\lambda+25$, so the eigenvalues of A are $\lambda=\dfrac{8\pm\sqrt{-36}}{2}=4\pm 3i.$

For $\lambda = 4+3i$: $A-(4+3i)I=\begin{bmatrix}-3i & 3\\ -3 & -3i\end{bmatrix}$. The equation $(A-(4+3i)I)\mathbf{x}=\mathbf{0}$ amounts to $x_1+ix_2=0$, so $x_1=-ix_2$ with x_2 free. A basis vector for the eigenspace is thus $\mathbf{v}_1=\begin{bmatrix}-i\\ 1\end{bmatrix}$.

For $\lambda = 4-3i$: A basis vector for the eigenspace is $\mathbf{v}_2=\overline{\mathbf{v}}_1=\begin{bmatrix}i\\ 1\end{bmatrix}$.

7. $A=\begin{bmatrix}\sqrt{3} & -1\\ 1 & \sqrt{3}\end{bmatrix}$. From Example 6, the eigenvalues are $\sqrt{3}\pm i$. The scale factor for the transformation $\mathbf{x}\mapsto A\mathbf{x}$ is $r=|\lambda|=\sqrt{(\sqrt{3})^2+1^2}=2$. For the angle of rotation, plot the point $(a,b)=(\sqrt{3},1)$ in the xy-plane and use trigonometry:

$$\varphi=\arctan(b/a)=\arctan(1/\sqrt{3})=\pi/6 \text{ radians.}$$



Note: Your students will want to know whether you permit them on an exam to omit calculations for a matrix of the form $\begin{bmatrix}a & -b\\ b & a\end{bmatrix}$ and simply write the eigenvalues $a\pm bi$. A similar question may arise about the corresponding eigenvectors, $\begin{bmatrix}1\\ -i\end{bmatrix}$ and $\begin{bmatrix}1\\ i\end{bmatrix}$, which are announced in the Practice Problem. Students may have trouble keeping track of the correspondence between eigenvalues and eigenvectors.

8. $A=\begin{bmatrix}\sqrt{3} & 3\\ -3 & \sqrt{3}\end{bmatrix}$. From Example 6, the eigenvalues are $\sqrt{3}\pm 3i$. The scale factor for the transformation $\mathbf{x}\mapsto A\mathbf{x}$ is $r=|\lambda|=\sqrt{(\sqrt{3})^2+3^2}=2\sqrt{3}$. From trigonometry, the angle of rotation φ is $\arctan(b/a)=\arctan(-3/\sqrt{3})=-\pi/3$ radians.

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9. $A=\begin{bmatrix}-\sqrt{3}/2 & 1/2\\ -1/2 & -\sqrt{3}/2\end{bmatrix}$. From Example 6, the eigenvalues are $-\sqrt{3}/2\pm(1/2)i$. The scale factor for the transformation $\mathbf{x}\mapsto A\mathbf{x}$ is $r=|\lambda|=\sqrt{(-\sqrt{3}/2)^2+(1/2)^2}=1$. From trigonometry, the angle of rotation φ is $\arctan(b/a)=\arctan((-1/2)/(-\sqrt{3}/2))=-5\pi/6$ radians.

10. $A=\begin{bmatrix}3 & 3\\ -3 & 3\end{bmatrix}$. From Example 6, the eigenvalues are $3\pm 3i$. The scale factor for the transformation $\mathbf{x}\mapsto A\mathbf{x}$ is $r=|\lambda|=\sqrt{(3)^2+3^2}=3\sqrt{2}$. From trigonometry, the angle of rotation φ is $\arctan(b/a)=\arctan(3/3)=-\pi/4$ radians.

11. $A=\begin{bmatrix}.1 & .1\\ -.1 & .1\end{bmatrix}$. From Example 6, the eigenvalues are $.1\pm .1i$. The scale factor for the transformation $\mathbf{x}\mapsto A\mathbf{x}$ is $r=|\lambda|=\sqrt{(.1)^2+(.1)^2}=\sqrt{2}/10$. From trigonometry, the angle of rotation φ is $\arctan(b/a)=\arctan(-.1/.1)=-\pi/4$ radians.

12. $A=\begin{bmatrix}0 & 4\\ -4 & 0\end{bmatrix}$. From Example 6, the eigenvalues are $0\pm 4i$. The scale factor for the transformation $\mathbf{x}\mapsto A\mathbf{x}$ is $r=|\lambda|=\sqrt{0^2+(4)^2}=4$. From trigonometry, the angle of rotation φ is $\arctan(b/a)=\arctan(-\infty)=-\pi/2$ radians.

13. From Exercise 1, $\lambda=2\pm i$, and the eigenvector $\mathbf{v}=\begin{bmatrix}-1-i\\ 1\end{bmatrix}$ corresponds to $\lambda=2-i$. Since Re $\mathbf{v}=\begin{bmatrix}-1\\ 1\end{bmatrix}$ and Im $\mathbf{v}=\begin{bmatrix}-1\\ 0\end{bmatrix}$, take $P=\begin{bmatrix}-1 & -1\\ 1 & 0\end{bmatrix}$. Then compute

$C=P^{-1}AP=\begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix}\begin{bmatrix}1 & -2\\ 1 & 3\end{bmatrix}\begin{bmatrix}-1 & -1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}0 & 1\\ -1 & -1\end{bmatrix}\begin{bmatrix}-3 & -1\\ 2 & -1\end{bmatrix}=\begin{bmatrix}2 & -1\\ 1 & 2\end{bmatrix}$. Actually, Theorem 9 gives the formula for C. Note that the eigenvector $\mathbf{v}$ corresponds to $a-bi$ instead of $a+bi$. If, for instance, you use the eigenvector for $2+i$, your C will be $\begin{bmatrix}2 & 1\\ -1 & 2\end{bmatrix}$.

Notes: The *Study Guide* points out that the matrix C is described in Theorem 9 and the first column of C is the real part of the eigenvector corresponding to $a-bi$, not $a+bi$, as one might expect. Since students may forget this, they are encouraged to compute C from the formula $C=P^{-1}AP$, as in the solution above.

The *Study Guide* also comments that because there are two possibilities for C in the factorization of a 2×2 matrix as in Exercise 13, the measure of rotation of the angle associated with the transformation $\mathbf{x}\mapsto A\mathbf{x}$ is determined only up to a change of sign. The "orientation" of the angle is determined by the change of variable $\mathbf{x}=P\mathbf{u}$. See Figure 4 in the text.

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14. $A = \begin{bmatrix} -1 & -1 \\ 5 & -5 \end{bmatrix}$. From Exercise 2, the eigenvalues of A are $\lambda = -3 \pm i$, and the eigenvector $\mathbf{v} = \begin{bmatrix} 2-i \\ 5 \end{bmatrix}$ corresponds to $\lambda = -3 - i$. By Theorem 9, $P = [\text{Re}\,\mathbf{v} \;\; \text{Im}\,\mathbf{v}] = \begin{bmatrix} 2 & -1 \\ 5 & 0 \end{bmatrix}$ and

$$C = P^{-1}AP = \frac{1}{5}\begin{bmatrix} 0 & 1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 5 & -5 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 1 & -3 \end{bmatrix}$$

15. $A = \begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix}$. From Exercise 3, the eigenvalues of A are $\lambda = 2 \pm 3i$, and the eigenvector $\mathbf{v} = \begin{bmatrix} 1+3i \\ 2 \end{bmatrix}$ corresponds to $\lambda = 2 - 3i$. By Theorem 9, $P = [\text{Re}\,\mathbf{v} \;\; \text{Im}\,\mathbf{v}] = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$ and

$$C = P^{-1}AP = -\frac{1}{6}\begin{bmatrix} 0 & -3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 5 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$$

16. $A = \begin{bmatrix} -3 & -1 \\ 2 & -5 \end{bmatrix}$. From Exercise 4, the eigenvalues of A are $\lambda = -4 \pm i$, and the eigenvector $\mathbf{v} = \begin{bmatrix} 1-i \\ 2 \end{bmatrix}$ corresponds to $\lambda = -4 - i$. By Theorem 9, $P = \left[\text{Re}\,\mathbf{v} \quad \text{Im}\,\mathbf{v}\right] = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$ and

$$C = P^{-1}AP = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -3 & -1 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -4 & -1 \\ 1 & -4 \end{bmatrix}.$$

17. $A = \begin{bmatrix} 1 & -.8 \\ 4 & -2.2 \end{bmatrix}$. The characteristic polynomial is $\lambda^2 + 1.2\lambda + 1$, so the eigenvalues of A are $\lambda = -.6 \pm .8i$. To find an eigenvector corresponding to $-.6 - .8i$, we compute $A - (-.6 - .8i)I = \begin{bmatrix} 1.6 + .8i & -.8 \\ 4 & -1.6 + .8i \end{bmatrix}$. The equation $(A - (-.6 - .8i)I)\mathbf{x} = \mathbf{0}$ amounts to $4x_1 + (-1.6 + .8i)x_2 = 0$, so $x_1 = ((2 - i)/5)x_2$ with x_2 free. A nice eigenvector corresponding to $-.6 - .8i$ is thus $\mathbf{v} = \begin{bmatrix} 2-i \\ 5 \end{bmatrix}$. By Theorem 9, $P = \left[\text{Re}\,\mathbf{v} \quad \text{Im}\,\mathbf{v}\right] = \begin{bmatrix} 2 & -1 \\ 5 & 0 \end{bmatrix}$ and

$$C = P^{-1}AP = \frac{1}{5}\begin{bmatrix} 0 & 1 \\ -5 & 2 \end{bmatrix}\begin{bmatrix} 1 & -.8 \\ 4 & -2.2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} -.6 & -.8 \\ .8 & -.6 \end{bmatrix}.$$

18. $A = \begin{bmatrix} 1 & -1 \\ .4 & .6 \end{bmatrix}$. The characteristic polynomial is $\lambda^2 - 1.6\lambda + 1$, so the eigenvalues of A are $\lambda = .8 \pm .6i$.

To find an eigenvector corresponding to $.8 - .6i$, we compute $A - (.8 - .6i)I = \begin{bmatrix} .2 + .6i & -1 \\ .4 & -.2 + .6i \end{bmatrix}$.

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The equation $(A-(.8-.6i)I)\mathbf{x}=\mathbf{0}$ amounts to $.4x_1+(-.2+.6i)x_2=0,$ so $x_1=((1-3i)/2)x_2$ with x_2 free. A nice eigenvector corresponding to $.8-.6i$ is thus $\mathbf{v}=\begin{bmatrix}1-3i\\2\end{bmatrix}$. By Theorem 9,

$$P=[\text{Re }\mathbf{v}\quad \text{Im }\mathbf{v}]=\begin{bmatrix}1&-3\\2&0\end{bmatrix} \text{ and } C=P^{-1}AP=\frac{1}{6}\begin{bmatrix}0&3\\-2&1\end{bmatrix}\begin{bmatrix}1&-1\\.4&.6\end{bmatrix}\begin{bmatrix}1&-3\\2&0\end{bmatrix}=\begin{bmatrix}.8&-.6\\.6&.8\end{bmatrix}.$$

19. $A=\begin{bmatrix}1.52&-.7\\.56&.4\end{bmatrix}$. The characteristic polynomial is $\lambda^2-1.92\lambda+1,$ so the eigenvalues of A are $\lambda=.96\pm.28i.$ To find an eigenvector corresponding to $.96-.28i,$ we compute $A-(.96-.28i)I=\begin{bmatrix}.56+.28i&-.7\\.56&-.56+.28i\end{bmatrix}$. The equation $(A-(.96-.28i)I)\mathbf{x}=\mathbf{0}$ amounts to $.56x_1+(-.56+.28i)x_2=0,$ so $x_1=((2-i)/2)x_2$ with x_2 free. A nice eigenvector corresponding to $.96-.28i$ is thus $\mathbf{v}=\begin{bmatrix}2-i\\2\end{bmatrix}$. By Theorem 9, $P=[\text{Re }\mathbf{v}\quad \text{Im }\mathbf{v}]=\begin{bmatrix}2&-1\\2&0\end{bmatrix}$ and

$$C=P^{-1}AP=\frac{1}{2}\begin{bmatrix}0&1\\-2&2\end{bmatrix}\begin{bmatrix}1.52&-.7\\.56&.4\end{bmatrix}\begin{bmatrix}2&-1\\2&0\end{bmatrix}=\begin{bmatrix}.96&-.28\\.28&.96\end{bmatrix}.$$

20. $A=\begin{bmatrix}-1.64&-2.4\\1.92&2.2\end{bmatrix}$. The characteristic polynomial is $\lambda^2-.56\lambda+1,$ so the eigenvalues of A are $\lambda=.28\pm.96i.$ To find an eigenvector corresponding to $.28-.96i,$ we compute $A-(.28-.96i)I=\begin{bmatrix}-1.92+.96i&-2.4\\1.92&1.92+.96i\end{bmatrix}$. The equation $(A-(.28-.96i)I)\mathbf{x}=\mathbf{0}$ amounts to $1.92x_1+(1.92+.96i)x_2=0,$ so $x_1=((-2-i)/2)x_2$ with x_2 free. A nice eigenvector corresponding to $.28-.96i$ is thus $\mathbf{v}=\begin{bmatrix}-2-i\\2\end{bmatrix}$. By Theorem 9, $P=[\text{Re }\mathbf{v}\quad \text{Im }\mathbf{v}]=\begin{bmatrix}-2&-1\\2&0\end{bmatrix}$ and

$$C=P^{-1}AP=\frac{1}{2}\begin{bmatrix}0&1\\-2&-2\end{bmatrix}\begin{bmatrix}-1.64&-2.4\\1.92&2.2\end{bmatrix}\begin{bmatrix}-2&-1\\2&0\end{bmatrix}=\begin{bmatrix}.28&-.96\\.96&.28\end{bmatrix}.$$

21. The first equation in (2) is $(-.3+.6i)x_1-.6x_2=0.$ We solve this for x_2 to find that $x_2=((-.3+.6i)/.6)x_1=((-1+2i)/2)x_1.$ Letting $x_1=2,$ we find that $\mathbf{y}=\begin{bmatrix}2\\-1+2i\end{bmatrix}$ is an eigenvector for the matrix A. Since $\mathbf{y}=\begin{bmatrix}2\\-1+2i\end{bmatrix}=\frac{-1+2i}{5}\begin{bmatrix}-2-4i\\5\end{bmatrix}=\frac{-1+2i}{5}\mathbf{v}_1$ the vector $\mathbf{y}$ is a complex multiple of the vector $\mathbf{v}_1$ used in Example 2.

22. Since $A(\mu\mathbf{x})=\mu(A\mathbf{x})=\mu(\lambda\mathbf{x})=\lambda(\mu\mathbf{x}), \mu\mathbf{x}$ is an eigenvector of A.

23. False. Complex eigenvalues come in pairs.

24. False. There is no assumption here that A has complex eigenvalues.

25. True. The characteristic polynomial has degree 2.

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26. False. There are no real eigenvectors.

27. (a) properties of conjugates and the fact that $\bar{\mathbf{x}}^T = \overline{\mathbf{x}^T}$

(b) $\overline{A\mathbf{x}} = A\bar{\mathbf{x}}$ and A is real

(c) $\mathbf{x}^T A\bar{\mathbf{x}}$ is a scalar and hence may be viewed as a 1×1 matrix

(d) properties of transposes

(e) $A^T = A$ and the definition of q

28. $\bar{\mathbf{x}}^T A\mathbf{x} = \bar{\mathbf{x}}^T(\lambda\mathbf{x}) = \lambda \cdot \bar{\mathbf{x}}^T\mathbf{x}$ because $\mathbf{x}$ is an eigenvector. It is easy to see that $\bar{\mathbf{x}}^T\mathbf{x}$ is real (and positive) because $\bar{z}z$ is nonnegative for every complex number z. Since $\bar{\mathbf{x}}^T A\mathbf{x}$ is real, by Exercise 27, so is λ. Next, write $\mathbf{x} = \mathbf{u} + i\mathbf{v}$, where $\mathbf{u}$ and $\mathbf{v}$ are real vectors. Then $A\mathbf{x} = A(\mathbf{u} + i\mathbf{v}) = A\mathbf{u} + iA\mathbf{v}$ and $\lambda\mathbf{x} = \lambda\mathbf{u} + i\lambda\mathbf{v}$. The real part of $A\mathbf{x}$ is $A\mathbf{u}$ because the entries in A, $\mathbf{u}$, and $\mathbf{v}$ are all real. The real part of $\lambda\mathbf{x}$ is $\lambda\mathbf{u}$ because λ and the entries in $\mathbf{u}$ and $\mathbf{v}$ are real. Since $A\mathbf{x}$ and $\lambda\mathbf{x}$ are equal, their real parts are equal, too. (Apply the corresponding statement about complex numbers to each entry of $A\mathbf{x}$.) Thus $A\mathbf{u} = \lambda\mathbf{u}$, which shows that the real part of $\mathbf{x}$ is an eigenvector of A.

29. Write $\mathbf{x} = \text{Re}\,\mathbf{x} + i(\text{Im}\,\mathbf{x})$, so that $A\mathbf{x} = A(\text{Re}\,\mathbf{x}) + iA(\text{Im}\,\mathbf{x})$. Since A is real, so are $A(\text{Re}\,\mathbf{x})$ and $A(\text{Im}\,\mathbf{x})$. Thus $A(\text{Re}\,\mathbf{x})$ is the real part of $A\mathbf{x}$ and $A(\text{Im}\,\mathbf{x})$ is the imaginary part of $A\mathbf{x}$.

30. **a**. If $\lambda = a - bi$, then $A\mathbf{v} = \lambda\mathbf{v} = (a - bi)(\text{Re}\,\mathbf{v} + i\ \text{Im}\,\mathbf{v}) = \underbrace{(a\ \text{Re}\,\mathbf{v} + b\ \text{Im}\,\mathbf{v})}_{\text{Re}\,A v} + i\underbrace{(a\ \text{Im}\,\mathbf{v} - b\ \text{Re}\,\mathbf{v})}_{\text{Im}\,A v}$. By Exercise 29,

$$A(\text{Re}\,\mathbf{v}) = \text{Re}\,A\mathbf{v} = a\ \text{Re}\,\mathbf{v} + b\ \text{Im}\,\mathbf{v}$$
$$A(\text{Im}\,\mathbf{v}) = \text{Im}\,A\mathbf{v} = -b\ \text{Re}\,\mathbf{v} + a\ \text{Im}\,\mathbf{v}$$

b. Let $P = [\text{Re}\,\mathbf{v} \quad \text{Im}\,\mathbf{v}]$. By (a), $A(\text{Re}\,\mathbf{v}) = P\begin{bmatrix} a \\ b \end{bmatrix}, A(\text{Im}\,\mathbf{v}) = P\begin{bmatrix} -b \\ a \end{bmatrix}$. So

$$AP = [A(\text{Re}\,\mathbf{v}) \quad A(\text{Im}\,\mathbf{v})] = \left[P\begin{bmatrix} a \\ b \end{bmatrix} \quad P\begin{bmatrix} -b \\ a \end{bmatrix} \right] = P\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = PC.$$

31. $A = \begin{bmatrix} .7 & 1.1 & 2.0 & 1.7 \\ -2.0 & -4.0 & -8.6 & -7.4 \\ 0 & -.5 & -1.0 & -1.0 \\ 1.0 & 2.8 & 6.0 & 5.3 \end{bmatrix}$. `v=eig(A) =(.2+.5i,.2-.5i,.3+.1i,.3-.1i)`

For $\lambda = .2 - .5i$, an eigenvector is

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`nulbasis(A-ev(2)*eye(4))` $= \begin{bmatrix} 0.5000 - 0.5000i \\ -2.0000 + 0.0000i \\ 0.0000 - 0.0000i \\ 1.0000 \end{bmatrix}$, so that

$\mathbf{v}_1 = \begin{bmatrix} .5-.5i \\ -2 \\ 0 \\ 1 \end{bmatrix}$.

For $\lambda = .3 - .1i$, an eigenvector is

`nulbasis(A-ev(4)*eye(4))` $= \begin{bmatrix} -0.5000 - 0.0000i \\ 0.0000 + 0.5000i \\ -0.7500 - 0.2500i \\ 1.0000 \end{bmatrix}$, so that $\mathbf{v}_2 = \begin{bmatrix} -.5 \\ .5i \\ -.75-.25i \\ 1 \end{bmatrix}$.

Hence by Theorem 9, $P = \begin{bmatrix} \text{Re}\,\mathbf{v}_1 & \text{Im}\,\mathbf{v}_1 & \text{Re}\,\mathbf{v}_2 & \text{Im}\,\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} .5 & -.5 & -.5 & 0 \\ -2 & 0 & 0 & .5 \\ 0 & 0 & -.75 & -.25 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ and

$C = \begin{bmatrix} .2 & -.5 & 0 & 0 \\ .5 & .2 & 0 & 0 \\ 0 & 0 & .3 & -.1 \\ 0 & 0 & .1 & .3 \end{bmatrix}$. Other choices are possible, but C must equal $P^{-1}AP$.

32. $A = \begin{bmatrix} -1.4 & -2.0 & -2.0 & -2.0 \\ -1.3 & -.8 & -.1 & -.6 \\ .3 & -1.9 & -1.6 & -1.4 \\ 2.0 & 3.3 & 2.3 & 2.6 \end{bmatrix}$. `ev=eig(A)=(-.4+i,-.4-i,-.2+.5i,-.2-.5i)`

For $\lambda = -.4 - i$, an eigenvector is

`nulbasis(A-ev(2)*eye(4))` $= \begin{bmatrix} -1.0000 - 1.0000i \\ -1.0000 + 1.0000i \\ 1.0000 - 1.0000i \\ 1.0000 \end{bmatrix}$, so that $\mathbf{v}_1 = \begin{bmatrix} -1-i \\ -1+i \\ 1-i \\ 1 \end{bmatrix}$.

For $\lambda = -.2 - .5i$, an eigenvector is

`nulbasis(A-ev(4)*eye(4))` $= \begin{bmatrix} 0.0000 - 0.0000i \\ -0.5000 - 0.5000i \\ -0.5000 + 0.5000i \\ 1.0000 \end{bmatrix}$, so that $\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1-i \\ -1+i \\ 2 \end{bmatrix}$

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Hence by Theorem 9, $P = \begin{bmatrix} \text{Re}\,\mathbf{v}_1 & \text{Im}\,\mathbf{v}_1 & \text{Re}\,\mathbf{v}_2 & \text{Im}\,\mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 & 0 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & 0 & 2 & 0 \end{bmatrix}$ and

$C = \begin{bmatrix} -.4 & -1 & 0 & 0 \\ 1 & -.4 & 0 & 0 \\ 0 & 0 & -.2 & -.5 \\ 0 & 0 & .5 & -.2 \end{bmatrix}$. Other choices are possible, but C must equal $P^{-1}AP$.

5.6 - Discrete Dynamical Systems

1. The exercise does not specify the matrix A, but only lists the eigenvalues 3 and 1/3, and the corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Also, $\mathbf{x}_0 = \begin{bmatrix} 9 \\ 1 \end{bmatrix}$.

a. To find the action of A on $\mathbf{x}_0$, express $\mathbf{x}_0$ in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$. That is, find c_1 and c_2 such that $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. This is certainly possible because the eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent (by inspection and also because they correspond to distinct eigenvalues) and hence form a basis for $\mathbf{R}^2$. (Two linearly independent vectors in $\mathbf{R}^2$ automatically span $\mathbf{R}^2$.) The row reduction $\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{x}_0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 9 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -4 \end{bmatrix}$ shows that $\mathbf{x}_0 = 5\mathbf{v}_1 - 4\mathbf{v}_2$. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors (for the eigenvalues 3 and 1/3):

$$\mathbf{x}_1 = A\mathbf{x}_0 = 5A\mathbf{v}_1 - 4A\mathbf{v}_2 = 5\cdot 3\mathbf{v}_1 - 4\cdot(1/3)\mathbf{v}_2 = \begin{bmatrix} 15 \\ 15 \end{bmatrix} - \begin{bmatrix} -4/3 \\ 4/3 \end{bmatrix} = \begin{bmatrix} 49/3 \\ 41/3 \end{bmatrix}$$

b. Each time A acts on a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$, the $\mathbf{v}_1$ term is multiplied by the eigenvalue 3 and the $\mathbf{v}_2$ term is multiplied by the eigenvalue 1/3:

$$\mathbf{x}_2 = A\mathbf{x}_1 = A[5\cdot 3\mathbf{v}_1 - 4(1/3)\mathbf{v}_2] = 5(3)^2\mathbf{v}_1 - 4(1/3)^2\mathbf{v}_2$$

In general, $\mathbf{x}_k = 5(3)^k\mathbf{v}_1 - 4(1/3)^k\mathbf{v}_2$, for $k \ge 0$.

2. The vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 \\ -3 \\ 7 \end{bmatrix}$ are eigenvectors of a 3×3 matrix A, corresponding to eigenvalues 3, 4/5, and 3/5, respectively. Also, $\mathbf{x}_0 = \begin{bmatrix} -2 \\ -5 \\ 3 \end{bmatrix}$. To describe the solution of the equation $\mathbf{x}_{k+1} = A\mathbf{x}_k\,(k = 1, 2, \ldots)$, first write $\mathbf{x}_0$ in terms of the eigenvectors.

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{x}_0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -3 & -2 \\ 0 & 1 & -3 & -5 \\ -3 & -5 & 7 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \Rightarrow \mathbf{x}_0 = 2\mathbf{v}_1 + \mathbf{v}_2 + 2\mathbf{v}_3$$

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Then, $\mathbf{x}_1 = A(2\mathbf{v}_1 + \mathbf{v}_2 + 2\mathbf{v}_3) = 2A\mathbf{v}_1 + A\mathbf{v}_2 + 2A\mathbf{v}_3 = 2\cdot 3\mathbf{v}_1 + (4/5)\mathbf{v}_2 + 2\cdot(3/5)\mathbf{v}_3$. In general, $\mathbf{x}_k = 2\cdot 3^k\mathbf{v}_1 + (4/5)^k\mathbf{v}_2 + 2\cdot(3/5)^k\mathbf{v}_3$. For all k sufficiently large,

$$\mathbf{x}_k \approx 2\cdot 3^k\mathbf{v}_1 = 2\cdot 3^k\begin{bmatrix}1\\0\\-3\end{bmatrix}.$$

3. $A = \begin{bmatrix} .5 & .4 \\ -.2 & 1.1\end{bmatrix}$, $\det(A-\lambda I) = (.5-\lambda)(1.1-\lambda) + .08 = \lambda^2 - 1.6\lambda + .63$. This characteristic polynomial factors as $(\lambda - .9)(\lambda - .7)$, so the eigenvalues are .9 and .7. If $\mathbf{v}_1$ and $\mathbf{v}_2$ denote corresponding eigenvectors, and if $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, then

$\mathbf{x}_1 = A(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) = c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 = c_1(.9)\mathbf{v}_1 + c_2(.7)\mathbf{v}_2$, and for $k \ge 1$,

$\mathbf{x}_k = c_1(.9)^k\mathbf{v}_1 + c_2(.7)^k\mathbf{v}_2$. For any choices of c_1 and c_2, both the owl and wood rat populations decline over time.

4. $A = \begin{bmatrix} .5 & .4 \\ -.125 & 1.1\end{bmatrix}$, $\det(A-\lambda I) = (.5-\lambda)(1.1-\lambda) - (.4)(-.125) = \lambda^2 - 1.6\lambda + .6$. This characteristic polynomial factors as $(\lambda - 1)(\lambda - .6)$, so the eigenvalues are 1 and .6. For the eigenvalue 1, solve $(A-I)\mathbf{x} = 0: \begin{bmatrix} -.5 & .4 & 0 \\ -.125 & .1 & 0\end{bmatrix} \sim \begin{bmatrix} -5 & 4 & 0 \\ 0 & 0 & 0\end{bmatrix}$. A basis for the eigenspace is $\mathbf{v}_1 = \begin{bmatrix}4\\5\end{bmatrix}$. Let $\mathbf{v}_2$ be an eigenvector for the eigenvalue .6. (The entries in $\mathbf{v}_2$ are not important for the long-term behavior of the system.). If $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, then $\mathbf{x}_1 = c_1A\mathbf{v}_1 + c_2A\mathbf{v}_2 = c_1\mathbf{v}_1 + c_2(.6)\mathbf{v}_2$, and for k sufficiently large, $\mathbf{x}_k = c_1\begin{bmatrix}4\\5\end{bmatrix} + c_2(.6)^k\mathbf{v}_2 \approx c_1\begin{bmatrix}4\\5\end{bmatrix}$. Provided that $c_1 \ne 0$, the owl and wood rat populations each stabilize in size, and eventually the populations are in the ratio of 4 owls for each 5 thousand rats. If some aspect of the model were to change slightly, the characteristic equation would change slightly and the perturbed matrix A might not have 1 as an eigenvalue. If the eigenvalue becomes slightly large than 1, the two populations will grow; if the eigenvalue becomes slightly less than 1, both populations will decline.

5. $A = \begin{bmatrix} .4 & .3 \\ -.325 & 1.2\end{bmatrix}$, $\det(A-\lambda I) = \lambda^2 - 1.6\lambda + .5775$. The quadratic formula provides the roots of the characteristic equation: $\lambda = \dfrac{1.6 \pm \sqrt{1.6^2 - 4(.5775)}}{2} = \dfrac{1.6 \pm \sqrt{.25}}{2} = 1.05 \text{ and } .55$.

Because one eigenvalue is larger than one, both populations grow in size. Their relative sizes are determined eventually by the entries in the eigenvector corresponding to 1.05. Solve $(A - 1.05I)\mathbf{x} = \mathbf{0}: \begin{bmatrix} -.65 & .3 & 0 \\ -.325 & .15 & 0\end{bmatrix} \sim \begin{bmatrix} -13 & 6 & 0 \\ 0 & 0 & 0\end{bmatrix}$. An eigenvector is $\mathbf{v}_1 = \begin{bmatrix}6\\13\end{bmatrix}$.

Eventually, there will be about 6 spotted owls for every 13 (thousand) flying squirrels.

6. When $p = .5$, $A = \begin{bmatrix} .4 & .3 \\ -.5 & 1.2\end{bmatrix}$, and $\det(A-\lambda I) = \lambda^2 - 1.6\lambda + .63 = (\lambda - .9)(\lambda - .7)$.

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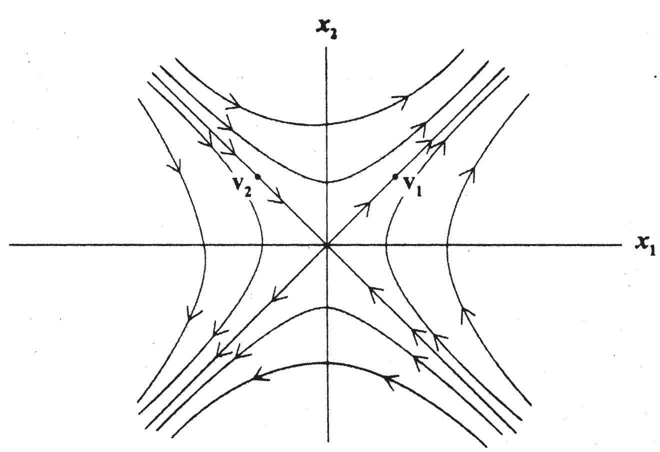
The eigenvalues of A are .9 and .7, both less than 1 in magnitude. The origin is an attractor for the dynamical system and each trajectory tends toward $\mathbf{0}$. So both populations of owls and squirrels eventually perish.
The calculations in Exercise 4 (as well as those in Exercises 43 and 35 in Section 5.1) show that if the largest eigenvalue of A is 1, then in most cases the population vector $\mathbf{x}_k$ will tend toward a multiple of the eigenvector corresponding to the eigenvalue 1. [If $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors, with $\mathbf{v}_1$ corresponding to $\lambda = 1$, and if $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, then $\mathbf{x}_k$ tends toward $c_1\mathbf{v}_1$, provided c_1 is not zero.] So the problem here is to determine the value of the predation parameter p such that the largest eigenvalue of A is 1. Compute the characteristic polynomial:
$\det\begin{bmatrix} .4-\lambda & .3 \\ -p & 1.2-\lambda \end{bmatrix} = (.4-\lambda)(1.2-\lambda) + .3p = \lambda^2 - 1.6\lambda + (.48 + .3p)$. By the quadratic formula, $\lambda = \dfrac{1.6 \pm \sqrt{1.6^2 - 4(.48+.3p)}}{2}$. The larger eigenvalue is 1 when
$1.6 + \sqrt{1.6^2 - 4(.48+.3p)} = 2$ and $\sqrt{2.56 - 1.92 - 1.2p} = .4$. In this case, $.64 - 1.2p = .16$, and $p = .4$.

7. **a**. The matrix A in Exercise 1 has eigenvalues 3 and 1/3. Since $|3| > 1$ and $|1/3| < 1$, the origin is a saddle point.

b. The direction of greatest attraction is determined by $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, the eigenvector corresponding to the eigenvalue with absolute value less than 1. The direction of greatest repulsion is determined by $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, the eigenvector corresponding to the eigenvalue greater than 1.

c. The drawing below shows: (1) lines through the eigenvectors and the origin, (2) arrows toward the origin (showing attraction) on the line through $\mathbf{v}_2$ and arrows away from the origin (showing repulsion) on the line through $\mathbf{v}_1$, (3) several typical trajectories (with arrows) that show the general flow of points. No specific points other than $\mathbf{v}_1$ and $\mathbf{v}_2$ were computed. This type of drawing is about all that one can make without using a computer to plot points.



Note: If you wish your class to sketch trajectories for anything except saddle points, you will need to go beyond the discussion in the text. The following remarks from the *Study Guide* are relevant.

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Sketching trajectories for a dynamical system in which the origin is an attractor or a repeller is more difficult than the sketch in Exercise 7. There has been no discussion of the direction in which the trajectories "bend" as they move toward or away from the origin. For instance, if you rotate Figure 1 of Section 5.6 through a quarter-turn and relabel the axes so that x_1 is on the horizontal axis, then the new figure corresponds to the matrix A with the diagonal entries .8 and .64 interchanged. In general, if A is a diagonal matrix, with positive diagonal entries a and d, unequal to 1, then the trajectories lie on the axes or on curves whose equations have the form $x_2 = r(x_1)^s$, where $s = (\ln d)/(\ln a)$ and r depends on the initial point $\mathbf{x}_0$. (See *Encounters with Chaos*, by Denny Gulick, New York: McGraw-Hill, 1992, pp. 147–150.)

8. The matrix from Exercise 2 has eigenvalues 3, 4/5, and 3/5. Since one eigenvalue is greater than 1 and the others are less than one in magnitude, the origin is a saddle point. The direction of greatest repulsion is the line through the origin and the eigenvector $(1,0,-3)$ for the eigenvalue 3. The direction of greatest attraction is the line through the origin and the eigenvector $(-3,-3,7)$ for the smallest eigenvalue 3/5.

9. $A = \begin{bmatrix} 1.7 & -.3 \\ -1.2 & .8 \end{bmatrix}$, $\det(A - \lambda I) = \lambda^2 - 2.5\lambda + 1 = 0$,

$\lambda = \dfrac{2.5 \pm \sqrt{2.5^2 - 4(1)}}{2} = \dfrac{2.5 \pm \sqrt{2.25}}{2} = \dfrac{2.5 \pm 1.5}{2} = 2 \text{ and } .5$. The origin is a saddle point because one eigenvalue is greater than 1 and the other eigenvalue is less than 1 in magnitude. The direction of greatest repulsion is through the origin and the eigenvector $\mathbf{v}_1$ found below. Solve $(A - 2I)\mathbf{x} = \mathbf{0}: \begin{bmatrix} -.3 & -.3 & 0 \\ -1.2 & -1.2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $x_1 = -x_2$, and x_2 is free. Take $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. The direction of greatest attraction is through the origin and the eigenvector $\mathbf{v}_2$ found below. Solve $(A - .5I)\mathbf{x} = \mathbf{0}: \begin{bmatrix} 1.2 & -.3 & 0 \\ -1.2 & .3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -.25 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $x_1 = -.25x_2$, and x_2 is free. Take $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

10. $A = \begin{bmatrix} .3 & .4 \\ -.3 & 1.1 \end{bmatrix}$, $\det(A - \lambda I) = \lambda^2 - 1.4\lambda + .45 = 0$,

$\lambda = \dfrac{1.4 \pm \sqrt{1.4^2 - 4(.45)}}{2} = \dfrac{1.4 \pm \sqrt{.16}}{2} = \dfrac{1.4 \pm .4}{2} = .5 \text{ and } .9$. The origin is an attractor because both eigenvalues are less than 1 in magnitude. The direction of greatest attraction is through the origin and the eigenvector $\mathbf{v}_1$ found below. Solve $(A - .5I)\mathbf{x} = \mathbf{0}: \begin{bmatrix} -.2 & .4 & 0 \\ -.3 & .6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $x_1 = 2x_2$, and x_2 is free. Take $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

11. $A = \begin{bmatrix} .4 & .5 \\ -.4 & 1.3 \end{bmatrix}$, $\det(A - \lambda I) = \lambda^2 - 1.7\lambda + .72 = 0$,

$\lambda = \dfrac{1.7 \pm \sqrt{1.7^2 - 4(.72)}}{2} = \dfrac{1.7 \pm \sqrt{.01}}{2} = \dfrac{1.7 \pm .1}{2} = .8 \text{ and } .9$. The origin is an attractor because both

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eigenvalues are less than 1 in magnitude. The direction of greatest attraction is through the origin and the eigenvector $\mathbf{v}_1$ found below. Solve $(A-.8I)\mathbf{x}=\mathbf{0}$: $\begin{bmatrix} -.4 & .5 & 0 \\ -.4 & .5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1.25 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $x_1 = 1.25x_2$, and x_2 is free. Take $\mathbf{v}_1 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$.

12. $A = \begin{bmatrix} .5 & .6 \\ -.3 & 1.4 \end{bmatrix}$, $\det(A-\lambda I) = \lambda^2 - 1.9\lambda + .88 = 0$.

$\lambda = \dfrac{1.9 \pm \sqrt{1.9^2 - 4(.88)}}{2} = \dfrac{1.9 \pm \sqrt{.09}}{2} = \dfrac{1.9 \pm .3}{2} = .8 \text{ and } 1.1$. The origin is a saddle point because one eigenvalue is greater than 1 and the other eigenvalue is less than 1 in magnitude. The direction of greatest repulsion is through the origin and the eigenvector $\mathbf{v}_1$ found below. Solve $(A-1.1I)\mathbf{x}=\mathbf{0}$: $\begin{bmatrix} -.6 & .6 & 0 \\ -.3 & .3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $x_1 = x_2$, and x_2 is free. Take $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The direction of greatest attraction is through the origin and the eigenvector $\mathbf{v}_2$ found below. Solve $(A-.8I)\mathbf{x}=\mathbf{0}$: $\begin{bmatrix} -.3 & .6 & 0 \\ -.3 & .6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $x_1 = 2x_2$, and x_2 is free. Take $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

13. $A = \begin{bmatrix} .8 & .3 \\ -.4 & 1.5 \end{bmatrix}$, $\det(A-\lambda I) = \lambda^2 - 2.3\lambda + 1.32 = 0$,

$\lambda = \dfrac{2.3 \pm \sqrt{2.3^2 - 4(1.32)}}{2} = \dfrac{2.3 \pm \sqrt{.01}}{2} = \dfrac{2.3 \pm .1}{2} = 1.1 \text{ and } 1.2$. The origin is a repeller because both eigenvalues are greater than 1 in magnitude. The direction of greatest repulsion is through the origin and the eigenvector $\mathbf{v}_1$ found below. Solve $(A-1.2I)\mathbf{x}=\mathbf{0}$: $\begin{bmatrix} -.4 & .3 & 0 \\ -.4 & .3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -.75 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $x_1 = .75x_2$, and x_2 is free. Take $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

14. $A = \begin{bmatrix} 1.7 & .6 \\ -.4 & .7 \end{bmatrix}$, $\det(A-\lambda I) = \lambda^2 - 2.4\lambda + 1.43 = 0$,

$\lambda = \dfrac{2.4 \pm \sqrt{2.4^2 - 4(1.43)}}{2} = \dfrac{2.4 \pm \sqrt{.04}}{2} = \dfrac{2.4 \pm .2}{2} = 1.1 \text{ and } 1.3$. The origin is a repeller because both eigenvalues are greater than 1 in magnitude. The direction of greatest repulsion is through the origin and the eigenvector $\mathbf{v}_1$ found below. Solve $(A-1.3I)\mathbf{x}=\mathbf{0}$: $\begin{bmatrix} .4 & .6 & 0 \\ -.4 & -.6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1.5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $x_1 = -1.5x_2$, and x_2 is free. Take $\mathbf{v}_1 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$.

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15. $A=\begin{bmatrix} .4 & 0 & .2 \\ .3 & .8 & .3 \\ .3 & .2 & .5 \end{bmatrix}$. Given eigenvector $\mathbf{v}_1=\begin{bmatrix} .1 \\ .6 \\ .3 \end{bmatrix}$ and eigenvalues .5 and .2. To find the eigenvalue for $\mathbf{v}_1$, compute $A\mathbf{v}_1=\begin{bmatrix} .4 & 0 & .2 \\ .3 & .8 & .3 \\ .3 & .2 & .5 \end{bmatrix}\begin{bmatrix} .1 \\ .6 \\ .3 \end{bmatrix}=\begin{bmatrix} .1 \\ .6 \\ .3 \end{bmatrix}=1\cdot\mathbf{v}_1$ Thus $\mathbf{v}_1$ is an eigenvector for $\lambda=1$.

For $\lambda=.5$: $\begin{bmatrix} -.1 & 0 & .2 & 0 \\ .3 & .3 & .3 & 0 \\ .3 & .2 & 0 & 0 \end{bmatrix}\sim\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{matrix} x_1=2x_3 \\ x_2=-3x_3 \\ x_3 \text{ is free} \end{matrix}$. Set $\mathbf{v}_2=\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$.

For $\lambda=.2$: $\begin{bmatrix} .2 & 0 & .2 & 0 \\ .3 & .6 & .3 & 0 \\ .3 & .2 & .3 & 0 \end{bmatrix}\sim\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\begin{matrix} x_1=-x_3 \\ x_2=0 \\ x_3 \text{ is free} \end{matrix}$. Set $\mathbf{v}_3=\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Given $\mathbf{x}_0=(0,.3,.7)$, find weights such that $\mathbf{x}_0=c_1\mathbf{v}_1+c\mathbf{v}_2+c_3\mathbf{v}_3$.

$$\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{x}_0\end{bmatrix}=\begin{bmatrix} .1 & 2 & -1 & 0 \\ .6 & -3 & 0 & .3 \\ .3 & 1 & 1 & .7 \end{bmatrix}\sim\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & .1 \\ 0 & 0 & 1 & .3 \end{bmatrix}.$$

$\mathbf{x}_0=\mathbf{v}_1+.1\mathbf{v}_2+.3\mathbf{v}_3$

$\mathbf{x}_1=A\mathbf{v}_1+.1A\mathbf{v}_2+.3A\mathbf{v}_3=\mathbf{v}_1+.1(.5)\mathbf{v}_2+.3(.2)\mathbf{v}_3$, and

$\mathbf{x}_k$=$\mathbf{v}_1$+.1$(.5)^k\,\mathbf{v}_2$+.3$(.2)^k\,\mathbf{v}_3$. As k increases, $\mathbf{x}_k$ approaches $\mathbf{v}_1$.

16.

$A=\begin{bmatrix} .90 & .01 & .09 \\ .01 & .90 & .01 \\ .09 & .09 & .90 \end{bmatrix}\cdot$ `ev = eig(A)` $=\begin{bmatrix} 1.0000 \\ 0.8900 \\ .8100 \end{bmatrix}$. To four decimal places,

$v_1 =$ `nulbasis(A-ev(1)*eye(3))` $=\begin{bmatrix} 0.9192 \\ 0.1919 \\ 1.0000 \end{bmatrix}$. Exact: $\begin{bmatrix} 91/99 \\ 19/99 \\ 1 \end{bmatrix}$

$v_2 =$ `nulbasis(A-ev(2)*eye(3))` $=\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$v_3 =$ `nulbasis(A-ev(3)*eye(3))` $=\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

The general solution of the dynamical system is $\mathbf{x}_k=c_1\mathbf{v}_1+c_2(.89)^k\,\mathbf{v}_2+c_3(.81)^k\,\mathbf{v}_3$.

Note: When working with stochastic matrices and starting with a probability vector (having nonnegative entries whose sum is 1), it helps to scale $\mathbf{v}_1$ to make its entries sum to 1. If $\mathbf{v}_1=(91/209, 19/209, 99/209)$, or $(.435, .091, .474)$ to three decimal places, then the weight c_1 above turns out to be 1.

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17. **a**. $A=\begin{bmatrix} 0 & 1.6 \\ .3 & .8 \end{bmatrix}$

b. $\det\begin{bmatrix} -\lambda & 1.6 \\ .3 & .8-\lambda \end{bmatrix}=\lambda^2-.8\lambda-.48=0.$ The eigenvalues of A are given by

$\lambda=\dfrac{.8\pm\sqrt{(-.8)^2-4(-.48)}}{2}=\dfrac{.8\pm\sqrt{2.56}}{2}=\dfrac{.8\pm1.6}{2}=1.2$ and $-.4$. The numbers of juveniles and adults are increasing because the largest eigenvalue is greater than 1. The eventual growth rate of each age class is 1.2, which is 20% per year. To find the eventual relative population sizes, solve $(A-1.2I)\mathbf{x}=\mathbf{0}$: $\begin{bmatrix} -1.2 & 1.6 & 0 \\ .3 & -.4 & 0 \end{bmatrix}\sim\begin{bmatrix} 1 & -4/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. $\begin{matrix} x_1=(4/3)x_2 \\ x_2 \text{ is free} \end{matrix}$. Set $\mathbf{v}_1=\begin{bmatrix} 4 \\ 3 \end{bmatrix}$.

Eventually, there will be about 4 juveniles for every 3 adults.

c. Suppose that the initial populations are given by $\mathbf{x}_0=(15,10)$. The *Study Guide* describes how to generate the trajectory for as many years as desired and then to plot the values for each population. Let $\mathbf{x}_k=(\mathrm{j}_k,\mathrm{a}_k)$. Then we need to plot the sequences $\{\mathrm{j}_k\}, \{\mathrm{a}_k\}, \{\mathrm{j}_k+\mathrm{a}_k\}$, and $\{\mathrm{j}_k/\mathrm{a}_k\}$. Adjacent points in a sequence can be connected with a line segment. When a sequence is plotted, the resulting graph can be captured on the screen and printed (if done on a computer) or copied by hand onto paper (if working with a graphics calculator).

18. **a**. $A=\begin{bmatrix} 0 & 0 & .42 \\ .6 & 0 & 0 \\ 0 & .75 & .95 \end{bmatrix}$

b. $\texttt{ev = eig(A)}=\begin{bmatrix} 0.0774+0.4063i \\ 0.0774-0.4063i \\ 1.1048 \end{bmatrix}$. The long-term growth rate is 1.105, about 10.5 % per year.

$\texttt{v = nulbasis(A - ev(3)*eye(3))}=\begin{bmatrix} 0.3801 \\ 0.2064 \\ 1.0000 \end{bmatrix}$. For each 100 adults, there will be approximately 38 calves and 21 yearlings.

Note: The MATLAB box in the *Study Guide* and the various technology appendices all give directions for generating the sequence of points in a trajectory of a dynamical system. Details for producing a graphical representation of a trajectory are also given, with several options available in MATLAB, Maple, and Mathematica.

5.7 - Applications to Differential Equations

1. From the "eigendata" (eigenvalues and corresponding eigenvectors) given, the eigenfunctions for the differential equation $\mathbf{x}'=A\mathbf{x}$ are $\mathbf{v}_1e^{4t}$ and $\mathbf{v}_2e^{2t}$. The general solution of $\mathbf{x}'=A\mathbf{x}$ has the form

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$c_1\begin{bmatrix}-3\\1\end{bmatrix}e^{4t}+c_2\begin{bmatrix}-1\\1\end{bmatrix}e^{2t}$. The initial condition $\mathbf{x}(0)=\begin{bmatrix}-6\\1\end{bmatrix}$ determines c_1 and c_2:

$c_1\begin{bmatrix}-3\\1\end{bmatrix}e^{4(0)}+c_2\begin{bmatrix}-1\\1\end{bmatrix}e^{2(0)}=\begin{bmatrix}-6\\1\end{bmatrix}$. Solving the system: $\begin{bmatrix}-3&-1&-6\\1&1&1\end{bmatrix}\sim\begin{bmatrix}1&0&5/2\\0&1&-3/2\end{bmatrix}$.

Thus $c_1=5/2, c_2=-3/2,$ and $\mathbf{x}(t)=\frac{5}{2}\begin{bmatrix}-3\\1\end{bmatrix}e^{4t}-\frac{3}{2}\begin{bmatrix}-1\\1\end{bmatrix}e^{2t}$.

2. From the eigendata given, the eigenfunctions for the differential equation $\mathbf{x}'=A\mathbf{x}$ are $\mathbf{v}_1e^{-3t}$ and $\mathbf{v}_2e^{-1t}$. The general solution of $\mathbf{x}'=A\mathbf{x}$ has the form $c_1\begin{bmatrix}-1\\1\end{bmatrix}e^{-3t}+c_2\begin{bmatrix}1\\1\end{bmatrix}e^{-1t}$. The initial condition $\mathbf{x}(0)=\begin{bmatrix}2\\3\end{bmatrix}$ determines c_1 and c_2: $c_1\begin{bmatrix}-1\\1\end{bmatrix}e^{-3(0)}+c_2\begin{bmatrix}1\\1\end{bmatrix}e^{-1(0)}=\begin{bmatrix}2\\3\end{bmatrix}$. Solving the system:

$\begin{bmatrix}-1&1&2\\1&1&3\end{bmatrix}\sim\begin{bmatrix}1&0&1/2\\0&1&5/2\end{bmatrix}$. Thus $c_1=1/2, c_2=5/2,$ and $\mathbf{x}(t)=\frac{1}{2}\begin{bmatrix}-1\\1\end{bmatrix}e^{-3t}+\frac{5}{2}\begin{bmatrix}1\\1\end{bmatrix}e^{-t}$.

3. $A=\begin{bmatrix}2&3\\-1&-2\end{bmatrix}$, $\det(A-\lambda I)=\lambda^2-1=(\lambda-1)(\lambda+1)=0$. Eigenvalues: 1 and -1.

<u>For λ = 1:</u> $\begin{bmatrix}1&3&0\\-1&-3&0\end{bmatrix}\sim\begin{bmatrix}1&3&0\\0&0&0\end{bmatrix}$, so $x_1=-3x_2$ with x_2 free. Take $x_2=1$ and $\mathbf{v}_1=\begin{bmatrix}-3\\1\end{bmatrix}$.

<u>For λ = –1:</u> $\begin{bmatrix}3&3&0\\-1&-1&0\end{bmatrix}\sim\begin{bmatrix}1&1&0\\0&0&0\end{bmatrix}$, so $x_1=-x_2$ with x_2 free. Take $x_2=1$ and $\mathbf{v}_2=\begin{bmatrix}-1\\1\end{bmatrix}$.

For the initial condition $\mathbf{x}(0)=\begin{bmatrix}3\\2\end{bmatrix}$, find c_1 and c_2 such that $c_1\mathbf{v}_1+c_2\mathbf{v}_2=\mathbf{x}(0)$:

$\begin{bmatrix}\mathbf{v}_1&\mathbf{v}_2&\mathbf{x}(0)\end{bmatrix}=\begin{bmatrix}-3&-1&3\\1&1&2\end{bmatrix}\sim\begin{bmatrix}1&0&-5/2\\0&1&9/2\end{bmatrix}$. Thus $c_1=-5/2, c_2=9/2,$ and

$\mathbf{x}(t)=-\frac{5}{2}\begin{bmatrix}-3\\1\end{bmatrix}e^{t}+\frac{9}{2}\begin{bmatrix}-1\\1\end{bmatrix}e^{-t}$.

Since one eigenvalue is positive and the other is negative, the origin is a saddle point of the dynamical system described by $\mathbf{x}'=A\mathbf{x}$. The direction of greatest attraction is the line through $\mathbf{v}_2$ and the origin. The direction of greatest repulsion is the line through $\mathbf{v}_1$ and the origin.

4. $A=\begin{bmatrix}-2&-5\\1&4\end{bmatrix}$, $\det(A-\lambda I)=\lambda^2-2\lambda-3=(\lambda+1)(\lambda-3)=0$. Eigenvalues: -1 and 3.

<u>For λ = 3:</u> $\begin{bmatrix}-5&-5&0\\1&1&0\end{bmatrix}\sim\begin{bmatrix}1&1&0\\0&0&0\end{bmatrix}$, so $x_1=-x_2$ with x_2 free. Take $x_2=1$ and $\mathbf{v}_1=\begin{bmatrix}-1\\1\end{bmatrix}$.

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For $\lambda = -1$: $\begin{bmatrix} -1 & -5 & 0 \\ 1 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $x_1 = -5x_2$ with x_2 free. Take $x_2 = 1$ and $\mathbf{v}_2 = \begin{bmatrix} -5 \\ 1 \end{bmatrix}$.

For the initial condition $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, find c_1 and c_2 such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{x}(0)$:

$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{x}(0) \end{bmatrix} = \begin{bmatrix} -1 & -5 & 3 \\ 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 13/4 \\ 0 & 1 & -5/4 \end{bmatrix}$. Thus $c_1 = 13/4, c_2 = -5/4$, and

$\mathbf{x}(t) = \frac{13}{4}\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{3t} - \frac{5}{4}\begin{bmatrix} -5 \\ 1 \end{bmatrix} e^{-t}$.

Since one eigenvalue is positive and the other is negative, the origin is a saddle point of the dynamical system described by $\mathbf{x}' = A\mathbf{x}$. The direction of greatest attraction is the line through $\mathbf{v}_2$ and the origin. The direction of greatest repulsion is the line through $\mathbf{v}_1$ and the origin.

5. $A = \begin{bmatrix} 7 & -1 \\ 3 & 3 \end{bmatrix}$, $\det(A - \lambda I) = \lambda^2 - 10\lambda + 24 = (\lambda - 4)(\lambda - 6) = 0$. Eigenvalues: 4 and 6.

For $\lambda = 4$: $\begin{bmatrix} 3 & -1 & 0 \\ 3 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $x_1 = (1/3)x_2$ with x_2 free. Take $x_2 = 3$ and $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

For $\lambda = 6$: $\begin{bmatrix} 1 & -1 & 0 \\ 3 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $x_1 = x_2$ with x_2 free. Take $x_2 = 1$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For the initial condition $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, find c_1 and c_2 such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{x}(0)$:

$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{x}(0) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 3 \\ 3 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 7/2 \end{bmatrix}$. Thus $c_1 = -1/2, c_2 = 7/2$, and

$\mathbf{x}(t) = -\frac{1}{2}\begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{4t} + \frac{7}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t}$.

Since both eigenvalues are positive, the origin is a repeller of the dynamical system described by $\mathbf{x}' = A\mathbf{x}$. The direction of greatest repulsion is the line through $\mathbf{v}_2$ and the origin.

6. $A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$, $\det(A - \lambda I) = \lambda^2 + 3\lambda + 2 = (\lambda + 1)(\lambda + 2) = 0$. Eigenvalues: -1 and -2.

For $\lambda = -2$: $\begin{bmatrix} 3 & -2 & 0 \\ 3 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $x_1 = (2/3)x_2$ with x_2 free. Take $x_2 = 3$ and $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

For $\lambda = -1$: $\begin{bmatrix} 2 & -2 & 0 \\ 3 & -3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so $x_1 = x_2$ with x_2 free. Take $x_2 = 1$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For the initial condition $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, find c_1 and c_2 such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{x}(0)$:

$[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{x}(0)] = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 5 \end{bmatrix}$. Thus $c_1 = -1, c_2 = 5$, and $\mathbf{x}(t) = -\begin{bmatrix} 2 \\ 3 \end{bmatrix} e^{-2t} + 5\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$.

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Since both eigenvalues are negative, the origin is an attractor of the dynamical system described by $\mathbf{x}' = A\mathbf{x}$. The direction of greatest attraction is the line through $\mathbf{v}_1$ and the origin.

7. From Exercise 5, $A = \begin{bmatrix} 7 & -1 \\ 3 & 3 \end{bmatrix}$, with eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ corresponding to eigenvalues 4 and 6 respectively. To decouple the equation $\mathbf{x}' = A\mathbf{x}$, set $P = [\mathbf{v}_1 \ \ \mathbf{v}_2] = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}$ and let $D = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix}$, so that $A = PDP^{-1}$ and $D = P^{-1}AP$. Substituting $\mathbf{x}(t) = P\mathbf{y}(t)$ into $\mathbf{x}' = A\mathbf{x}$ we have $\frac{d}{dt}(P\mathbf{y}) = A(P\mathbf{y}) = PDP^{-1}(P\mathbf{y}) = PD\mathbf{y}$. Since P has constant entries, $\frac{d}{dt}(P\mathbf{y}) = P(\frac{d}{dt}(\mathbf{y}))$, so that left-multiplying the equality $P(\frac{d}{dt}(\mathbf{y})) = PD\mathbf{y}$ by P^{-1} yields $\mathbf{y}' = D\mathbf{y}$, or $\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$.

8. From Exercise 6, $A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}$, with eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ corresponding to eigenvalues -2 and -1 respectively. To decouple the equation $\mathbf{x}' = A\mathbf{x}$, set $P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$ and let $D = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$, so that $A = PDP^{-1}$ and $D = P^{-1}AP$. Substituting $\mathbf{x}(t) = P\mathbf{y}(t)$ into $\mathbf{x}' = A\mathbf{x}$ we have $\frac{d}{dt}(P\mathbf{y}) = A(P\mathbf{y}) = PDP^{-1}(P\mathbf{y}) = PD\mathbf{y}$. Since P has constant entries, $\frac{d}{dt}(P\mathbf{y}) = P\left(\frac{d}{dt}(\mathbf{y})\right)$, so that left-multiplying the equality $P\left(\frac{d}{dt}(\mathbf{y})\right) = PD\mathbf{y}$ by P^{-1} yields $\mathbf{y}' = D\mathbf{y}$, or $\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$.

9. $A = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix}$. An eigenvalue of A is $-2 + i$ with corresponding eigenvector $\mathbf{v} = \begin{bmatrix} 1 - i \\ 1 \end{bmatrix}$. The complex eigenfunctions $\mathbf{v}e^{\lambda t}$ and $\overline{\mathbf{v}}e^{\overline{\lambda} t}$ form a basis for the set of all complex solutions to $\mathbf{x}' = A\mathbf{x}$. The general complex solution is $c_1 \begin{bmatrix} 1 - i \\ 1 \end{bmatrix} e^{(-2+i)t} + c_2 \begin{bmatrix} 1 + i \\ 1 \end{bmatrix} e^{(-2-i)t}$, where c_1 and c_2 are arbitrary complex numbers. To build the general real solution, rewrite $\mathbf{v}e^{(-2+i)t}$ as:

$$\begin{aligned} \mathbf{v}e^{(-2+i)t} &= \begin{bmatrix} 1 - i \\ 1 \end{bmatrix} e^{-2t} e^{it} = \begin{bmatrix} 1 - i \\ 1 \end{bmatrix} e^{-2t} (\cos t + i \sin t) \\ &= \begin{bmatrix} \cos t - i \cos t + i \sin t - i^2 \sin t \\ \cos t + i \sin t \end{bmatrix} e^{-2t} \\ &= \begin{bmatrix} \cos t + \sin t \\ \cos t \end{bmatrix} e^{-2t} + i \begin{bmatrix} \sin t - \cos t \\ \sin t \end{bmatrix} e^{-2t} \end{aligned}$$

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The general real solution has the form $c_1\begin{bmatrix}\cos t+\sin t\\ \cos t\end{bmatrix}e^{-2t}+c_2\begin{bmatrix}\sin t-\cos t\\ \sin t\end{bmatrix}e^{-2t}$, where c_1 and c_2 now are real numbers. The trajectories are spirals because the eigenvalues are complex. The spirals tend toward the origin because the real parts of the eigenvalues are negative.

10. $A=\begin{bmatrix}3 & 1\\ -2 & 1\end{bmatrix}$. An eigenvalue of A is $2+i$ with corresponding eigenvector $\mathbf{v}=\begin{bmatrix}1+i\\ -2\end{bmatrix}$. The complex eigenfunctions $\mathbf{v}e^{\lambda t}$ and $\overline{\mathbf{v}}e^{\overline{\lambda}t}$ form a basis for the set of all complex solutions to $\mathbf{x}'=A\mathbf{x}$. The general complex solution is $c_1\begin{bmatrix}1+i\\ -2\end{bmatrix}e^{(2+i)t}+c_2\begin{bmatrix}1-i\\ -2\end{bmatrix}e^{(2-i)t}$, where c_1 and c_2 are arbitrary complex numbers. To build the general real solution, rewrite $\mathbf{v}e^{(2+i)t}$ as:

$$\begin{aligned}\mathbf{v}e^{(2+i)t}&=\begin{bmatrix}1+i\\ -2\end{bmatrix}e^{2t}e^{it}=\begin{bmatrix}1+i\\ -2\end{bmatrix}e^{2t}(\cos t+i\sin t)\\ &=\begin{bmatrix}\cos t+i\cos t+i\sin t+i^2\sin t\\ -2\cos t-2i\sin t\end{bmatrix}e^{2t}\\ &=\begin{bmatrix}\cos t-\sin t\\ -2\cos t\end{bmatrix}e^{2t}+i\begin{bmatrix}\sin t+\cos t\\ -2\sin t\end{bmatrix}e^{2t}\end{aligned}$$

The general real solution has the form $c_1\begin{bmatrix}\cos t-\sin t\\ -2\cos t\end{bmatrix}e^{2t}+c_2\begin{bmatrix}\sin t+\cos t\\ -2\sin t\end{bmatrix}e^{2t}$, where c_1 and c_2 now are real numbers. The trajectories are spirals because the eigenvalues are complex. The spirals tend away from the origin because the real parts of the eigenvalues are positive.

11. $A=\begin{bmatrix}-3 & -9\\ 2 & 3\end{bmatrix}$. An eigenvalue of A is $3i$ with corresponding eigenvector $\mathbf{v}=\begin{bmatrix}-3+3i\\ 2\end{bmatrix}$. The complex eigenfunctions $\mathbf{v}e^{\lambda t}$ and $\overline{\mathbf{v}}e^{\overline{\lambda}t}$ form a basis for the set of all complex solutions to $\mathbf{x}'=A\mathbf{x}$. The general complex solution is $c_1\begin{bmatrix}-3+3i\\ 2\end{bmatrix}e^{(3i)t}+c_2\begin{bmatrix}-3-3i\\ 2\end{bmatrix}e^{(-3i)t}$, where c_1 and c_2 are arbitrary complex numbers. To build the general real solution, rewrite $\mathbf{v}e^{(3i)t}$ as:

$$\begin{aligned}\mathbf{v}e^{(3i)t}&=\begin{bmatrix}-3+3i\\ 2\end{bmatrix}(\cos 3t+i\sin 3t)\\ &=\begin{bmatrix}-3\cos 3t-3\sin 3t\\ 2\cos 3t\end{bmatrix}+i\begin{bmatrix}-3\sin 3t+3\cos 3t\\ 2\sin 3t\end{bmatrix}\end{aligned}$$

The general real solution has the form $c_1\begin{bmatrix}-3\cos 3t-3\sin 3t\\ 2\cos 3t\end{bmatrix}+c_2\begin{bmatrix}-3\sin 3t+3\cos 3t\\ 2\sin 3t\end{bmatrix}$, where c_1 and c_2 now are real numbers. The trajectories are ellipses about the origin because the real parts of the eigenvalues are zero.

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12. $A = \begin{bmatrix} -7 & 10 \\ -4 & 5 \end{bmatrix}$. An eigenvalue of A is $-1+2i$ with corresponding eigenvector $\mathbf{v} = \begin{bmatrix} 3-i \\ 2 \end{bmatrix}$. The complex eigenfunctions $\mathbf{v}e^{\lambda t}$ and $\overline{\mathbf{v}}e^{\overline{\lambda}t}$ form a basis for the set of all complex solutions to $\mathbf{x}' = A\mathbf{x}$. The general complex solution is $c_1 \begin{bmatrix} 3-i \\ 2 \end{bmatrix} e^{(-1+2i)t} + c_2 \begin{bmatrix} 3+i \\ 2 \end{bmatrix} e^{(-1-2i)t}$, where c_1 and c_2 are arbitrary complex numbers. To build the general real solution, rewrite $\mathbf{v}e^{(-1+2i)t}$ as:

$$\begin{aligned} \mathbf{v}e^{(-1+2i)t} &= \begin{bmatrix} 3-i \\ 2 \end{bmatrix} e^{-t}(\cos 2t + i \sin 2t) \\ &= \begin{bmatrix} 3\cos 2t + \sin 2t \\ 2\cos 2t \end{bmatrix} e^{-t} + i \begin{bmatrix} 3\sin 2t - \cos 2t \\ 2\sin 2t \end{bmatrix} e^{-t} \end{aligned}$$

The general real solution has the form $c_1 \begin{bmatrix} 3\cos 2t + \sin 2t \\ 2\cos 2t \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 3\sin 2t - \cos 2t \\ 2\sin 2t \end{bmatrix} e^{-t}$, where c_1 and c_2 now are real numbers. The trajectories are spirals because the eigenvalues are complex. The spirals tend toward the origin because the real parts of the eigenvalues are negative.

13. $A = \begin{bmatrix} 4 & -3 \\ 6 & -2 \end{bmatrix}$. An eigenvalue of A is $1+3i$ with corresponding eigenvector $\mathbf{v} = \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$. The complex eigenfunctions $\mathbf{v}e^{\lambda t}$ and $\overline{\mathbf{v}}e^{\overline{\lambda}t}$ form a basis for the set of all complex solutions to $\mathbf{x}' = A\mathbf{x}$. The general complex solution is $c_1 \begin{bmatrix} 1+i \\ 2 \end{bmatrix} e^{(1+3i)t} + c_2 \begin{bmatrix} 1-i \\ 2 \end{bmatrix} e^{(1-3i)t}$, where c_1 and c_2 are arbitrary complex numbers. To build the general real solution, rewrite $\mathbf{v}e^{(1+3i)t}$ as:

$$\begin{aligned} \mathbf{v}e^{(1+3i)t} &= \begin{bmatrix} 1+i \\ 2 \end{bmatrix} e^{t}(\cos 3t + i \sin 3t) \\ &= \begin{bmatrix} \cos 3t - \sin 3t \\ 2\cos 3t \end{bmatrix} e^{t} + i \begin{bmatrix} \sin 3t + \cos 3t \\ 2\sin 3t \end{bmatrix} e^{t} \end{aligned}$$

The general real solution has the form $c_1 \begin{bmatrix} \cos 3t - \sin 3t \\ 2\cos 3t \end{bmatrix} e^{t} + c_2 \begin{bmatrix} \sin 3t + \cos 3t \\ 2\sin 3t \end{bmatrix} e^{t}$, where c_1 and c_2 now are real numbers. The trajectories are spirals because the eigenvalues are complex. The spirals tend away from the origin because the real parts of the eigenvalues are positive.

14. $A = \begin{bmatrix} -2 & 1 \\ -8 & 2 \end{bmatrix}$. An eigenvalue of A is $2i$ with corresponding eigenvector $\mathbf{v} = \begin{bmatrix} 1-i \\ 4 \end{bmatrix}$. The complex eigenfunctions $\mathbf{v}e^{\lambda t}$ and $\overline{\mathbf{v}}e^{\overline{\lambda}t}$ form a basis for the set of all complex solutions to $\mathbf{x}' = A\mathbf{x}$. The general complex solution is $c_1 \begin{bmatrix} 1-i \\ 4 \end{bmatrix} e^{(2i)t} + c_2 \begin{bmatrix} 1+i \\ 4 \end{bmatrix} e^{(-2i)t}$, where c_1 and c_2 are arbitrary complex numbers. To build the general real solution, rewrite $\mathbf{v}e^{(2i)t}$ as:

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$$\mathbf{v}e^{(2i)t} = \begin{bmatrix} 1-i \\ 4 \end{bmatrix}(\cos 2t + i\sin 2t)$$
$$= \begin{bmatrix} \cos 2t + \sin 2t \\ 4\cos 2t \end{bmatrix} + i\begin{bmatrix} \sin 2t - \cos 2t \\ 4\sin 2t \end{bmatrix}$$

The general real solution has the form $c_1\begin{bmatrix} \cos 2t + \sin 2t \\ 4\cos 2t \end{bmatrix} + c_2\begin{bmatrix} \sin 2t - \cos 2t \\ 4\sin 2t \end{bmatrix}$, where c_1 and c_2 now are real numbers. The trajectories are ellipses about the origin because the real parts of the eigenvalues are zero.

15. $A = \begin{bmatrix} -8 & -12 & -6 \\ 2 & 1 & 2 \\ 7 & 12 & 5 \end{bmatrix}$. The eigenvalues of A are: `ev = eig(A)` = $\begin{bmatrix} 1.0000 \\ -1.0000 \\ -2.0000 \end{bmatrix}$.

`nulbasis(A-ev(1)*eye(3))` = $\begin{bmatrix} -1.0000 \\ 0.2500 \\ 1.0000 \end{bmatrix}$, so that $\mathbf{v}_1 = \begin{bmatrix} -4 \\ 1 \\ 4 \end{bmatrix}$.

`nulbasis(A-ev(2)*eye(3))` = $\begin{bmatrix} -1.2000 \\ 0.2000 \\ 1.0000 \end{bmatrix}$, so that $\mathbf{v}_2 = \begin{bmatrix} -6 \\ 1 \\ 5 \end{bmatrix}$.

`nulbasis (A-ev(3)*eye(3))` = $\begin{bmatrix} -1.0000 \\ 0.0000 \\ -1.0000 \end{bmatrix}$, so that $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Hence the general solution is $\mathbf{x}(t) = c_1\begin{bmatrix} -4 \\ 1 \\ 4 \end{bmatrix}e^{t} + c_2\begin{bmatrix} -6 \\ 1 \\ 5 \end{bmatrix}e^{-t} + c_3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}e^{-2t}$. The origin is a saddle point.

A solution with $c_1 = 0$ is attracted to the origin while a solution with $c_2 = c_3 = 0$ is repelled.

16. $A = \begin{bmatrix} -6 & -11 & 16 \\ 2 & 5 & -4 \\ -4 & -5 & 10 \end{bmatrix}$. The eigenvalues of A are: `ev = eig(A)` = $\begin{bmatrix} 4.0000 \\ 3.0000 \\ 2.0000 \end{bmatrix}$.

`nulbasis(A-ev(1)*eye(3))` = $\begin{bmatrix} 2.3333 \\ -0.6667 \\ 1.0000 \end{bmatrix}$, so that $\mathbf{v}_1 = \begin{bmatrix} 7 \\ -2 \\ 3 \end{bmatrix}$.

`nulbasis(A-ev(2)*eye(3))` = $\begin{bmatrix} 3.0000 \\ -1.0000 \\ 1.0000 \end{bmatrix}$, so that $\mathbf{v}_2 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$.

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`nulbasis(A-ev(3)*eye(3))` $= \begin{bmatrix} 2.0000 \\ 0.0000 \\ 1.0000 \end{bmatrix}$, so that $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$.

Hence the general solution is $\mathbf{x}(t) = c_1 \begin{bmatrix} 7 \\ -2 \\ 3 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} e^{2t}$. The origin is a repeller, because all eigenvalues are positive. All trajectories tend away from the origin.

17. $A = \begin{bmatrix} 30 & 64 & 23 \\ -11 & -23 & -9 \\ 6 & 15 & 4 \end{bmatrix}$. The eigenvalues of A are: `ev = eig(A)` $= \begin{bmatrix} \texttt{5.0000 + 2.0000i} \\ \texttt{5.0000 - 2.0000i} \\ \texttt{1.0000} \end{bmatrix}$.

`nulbasis(A-ev(1)*eye(3))` $= \begin{bmatrix} \texttt{7.6667 - 11.3333i} \\ \texttt{-3.0000 + 4.6667i} \\ \texttt{1.0000} \end{bmatrix}$, so that $\mathbf{v}_1 = \begin{bmatrix} 23-34i \\ -9+14i \\ 3 \end{bmatrix}$.

`nulbasis (A-ev(2)*eye(3))` $= \begin{bmatrix} \texttt{7.6667 + 11.3333i} \\ \texttt{-3.0000 - 4.6667i} \\ \texttt{1.0000} \end{bmatrix}$, so that $\mathbf{v}_2 = \begin{bmatrix} 23+34i \\ -9-14i \\ 3 \end{bmatrix}$.

`nulbasis (A-ev(3)*eye(3))` $= \begin{bmatrix} -3.0000 \\ 1.0000 \\ 1.0000 \end{bmatrix}$, so that $\mathbf{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$.

Hence the general complex solution is $\mathbf{x}(t) = c_1 \begin{bmatrix} 23-34i \\ -9+14i \\ 3 \end{bmatrix} e^{(5+2i)t} + c_2 \begin{bmatrix} 23+34i \\ -9-14i \\ 3 \end{bmatrix} e^{(5-2i)t} + c_3 \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} e^{t}$.

Rewriting the first eigenfunction yields

$$\begin{bmatrix} 23-34i \\ -9+14i \\ 3 \end{bmatrix} e^{5t}(\cos 2t + i\sin 2t) = \begin{bmatrix} 23\cos 2t + 34\sin 2t \\ -9\cos 2t - 14\sin 2t \\ 3\cos 2t \end{bmatrix} e^{5t} + i \begin{bmatrix} 23\sin 2t - 34\cos 2t \\ -9\sin 2t + 14\cos 2t \\ 3\sin 2t \end{bmatrix} e^{5t}.$$

Hence the general real solution is

$\mathbf{x}(t) = c_1 \begin{bmatrix} 23\cos 2t + 34\sin 2t \\ -9\cos 2t - 14\sin 2t \\ 3\cos 2t \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} 23\sin 2t - 34\cos 2t \\ -9\sin 2t + 14\cos 2t \\ 3\sin 2t \end{bmatrix} e^{5t} + c_3 \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} e^{t}$, where $c_1, c_2,$ and c_3 are real. The origin is a repeller, because the real parts of all eigenvalues are positive. All trajectories spiral away from the origin.

18. $A = \begin{bmatrix} 53 & -30 & -2 \\ 90 & -52 & -3 \\ 20 & -10 & 2 \end{bmatrix}$. The eigenvalues of A are: `ev = eig(A)` $= \begin{bmatrix} \texttt{-7.0000} \\ \texttt{5.0000 + 1.0000i} \\ \texttt{5.0000 - 1.0000i} \end{bmatrix}$.

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$$\texttt{nulbasis(A-ev(1)*eye(3))} = \begin{bmatrix} 0.5000 \\ 1.0000 \\ 0.0000 \end{bmatrix}, \text{ so that } \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

$$\texttt{nulbasis(A-ev(2)*eye(3))} = \begin{bmatrix} \texttt{0.6000 + 0.2000i} \\ \texttt{0.9000 + 0.3000i} \\ \texttt{1.0000} \end{bmatrix}, \text{ so that } \mathbf{v}_2 = \begin{bmatrix} 6+2i \\ 9+3i \\ 10 \end{bmatrix}.$$

$$\texttt{nulbasis(A-ev(3)*eye(3))} = \begin{bmatrix} \texttt{0.6000 - 0.20000} \\ \texttt{0.9000 - 0.3000i} \\ \texttt{1.0000} \end{bmatrix}, \quad \text{so that } \mathbf{v}_3 = \begin{bmatrix} 6-2i \\ 9-3i \\ 10 \end{bmatrix}.$$

Hence the general complex solution is $\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} e^{-7t} + c_2 \begin{bmatrix} 6+2i \\ 9+3i \\ 10 \end{bmatrix} e^{(5+i)t} + c_3 \begin{bmatrix} 6-2i \\ 9-3i \\ 10 \end{bmatrix} e^{(5-i)t}$.

Rewriting the second eigenfunction yields

$$\begin{bmatrix} 6+2i \\ 9+3i \\ 10 \end{bmatrix} e^{5t}(\cos t + i \sin t) = \begin{bmatrix} 6\cos t - 2\sin t \\ 9\cos t - 3\sin t \\ 10\cos t \end{bmatrix} e^{5t} + i \begin{bmatrix} 6\sin t + 2\cos t \\ 9\sin t + 3\cos t \\ 10\sin t \end{bmatrix} e^{5t}.$$

Hence the general real solution is $\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} e^{-7t} + c_2 \begin{bmatrix} 6\cos t - 2\sin t \\ 9\cos t - 3\sin t \\ 10\cos t \end{bmatrix} e^{5t} + c_3 \begin{bmatrix} 6\sin t + 2\cos t \\ 9\sin t + 3\cos t \\ 10\sin t \end{bmatrix} e^{5t}$,

where $c_1, c_2,$ and c_3 are real. When $c_2 = c_3 = 0$ the trajectories tend toward the origin, and in other cases the trajectories spiral away from the origin.

19. Substitute $R_1 = 1/5, R_2 = 1/3, C_1 = 4,$ and $C_2 = 3$ into the formula for A given in Example 1, and use a matrix program to find the eigenvalues and eigenvectors: $A = \begin{bmatrix} -2 & 3/4 \\ 1 & -1 \end{bmatrix}$, $\quad \lambda_1 = -.5 : \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$,

$\lambda_2 = -2.5 : \mathbf{v}_2 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. The general solution is thus $\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-.5t} + c_2 \begin{bmatrix} -3 \\ 2 \end{bmatrix} e^{-2.5t}$. The condition

$\mathbf{x}(0) = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ implies that $\begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$. By a matrix program, $c_1 = 5/2$ and $c_2 = -1/2$, so that

$$\begin{bmatrix} v_1(t) \\ v_2(t) \end{bmatrix} = \mathbf{x}(t) = \frac{5}{2} \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-.5t} - \frac{1}{2} \begin{bmatrix} -3 \\ 2 \end{bmatrix} e^{-2.5t}.$$

20. Substitute $R_1 = 1/15,\ R_2 = 1/3,\ C_1 = 9$ and $C_2 = 2$ into the formula for A given in Example 1, and use a matrix program to find the eigenvalues and eigenvectors: $A = \begin{bmatrix} -2 & 1/3 \\ 3/2 & -3/2 \end{bmatrix}$, .

$\lambda_1 = -1 : \mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\lambda_2 = -2.5 : \mathbf{v}_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. The general solution is thus

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$\mathbf{x}(t) = c_1\begin{bmatrix}1\\3\end{bmatrix}e^{-t} + c_2\begin{bmatrix}-2\\3\end{bmatrix}e^{-2.5t}$. The condition $\mathbf{x}(0) = \begin{bmatrix}3\\3\end{bmatrix}$ implies that $\begin{bmatrix}1 & -2\\3 & 3\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}3\\3\end{bmatrix}$. By a matrix program, $c_1 = 5/3$ and $c_2 = -2/3$, so that $\begin{bmatrix}v_1(t)\\v_2(t)\end{bmatrix} = \mathbf{x}(t) = \frac{5}{3}\begin{bmatrix}1\\3\end{bmatrix}e^{-t} - \frac{2}{3}\begin{bmatrix}-2\\3\end{bmatrix}e^{-2.5t}$.

21. $A = \begin{bmatrix}-1 & -8\\5 & -5\end{bmatrix}$. Using a matrix program we find that an eigenvalue of A is $-3+6i$ with corresponding eigenvector $\mathbf{v} = \begin{bmatrix}2+6i\\5\end{bmatrix}$. The conjugates of these form the second eigenvalue-eigenvector pair. The general complex solution is

$\mathbf{x}(t) = c_1\begin{bmatrix}2+6i\\5\end{bmatrix}e^{(-3+6i)t} + c_2\begin{bmatrix}2-6i\\5\end{bmatrix}e^{(-3-6i)t}$, where c_1 and c_2 are arbitrary complex numbers.

Rewriting the first eigenfunction and taking its real and imaginary parts, we have

$$\begin{aligned}\mathbf{v}e^{(-3+6i)t} &= \begin{bmatrix}2+6i\\5\end{bmatrix}e^{-3t}(\cos 6t + i\sin 6t)\\ &= \begin{bmatrix}2\cos 6t - 6\sin 6t\\5\cos 6t\end{bmatrix}e^{-3t} + i\begin{bmatrix}2\sin 6t + 6\cos 6t\\5\sin 6t\end{bmatrix}e^{-3t}\end{aligned}$$

The general real solution has the form $\mathbf{x}(t) = c_1\begin{bmatrix}2\cos 6t - 6\sin 6t\\5\cos 6t\end{bmatrix}e^{-3t} + c_2\begin{bmatrix}2\sin 6t + 6\cos 6t\\5\sin 6t\end{bmatrix}e^{-3t}$,

where c_1 and c_2 now are real numbers. To satisfy the initial condition $\mathbf{x}(0) = \begin{bmatrix}0\\15\end{bmatrix}$, we solve

$c_1\begin{bmatrix}2\\5\end{bmatrix} + c_2\begin{bmatrix}6\\0\end{bmatrix} = \begin{bmatrix}0\\15\end{bmatrix}$ to get $c_1 = 3, c_2 = -1$. We now have

$$\begin{bmatrix}i_L(t)\\v_C(t)\end{bmatrix} = \mathbf{x}(t) = 3\begin{bmatrix}2\cos 6t - 6\sin 6t\\5\cos 6t\end{bmatrix}e^{-3t} - \begin{bmatrix}2\sin 6t + 6\cos 6t\\5\sin 6t\end{bmatrix}e^{-3t} = \begin{bmatrix}-20\sin 6t\\15\cos 6t - 5\sin 6t\end{bmatrix}e^{-3t}.$$

22. $A = \begin{bmatrix}0 & 2\\-.4 & -.8\end{bmatrix}$. Using a matrix program we find that an eigenvalue of A is $-.4+.8i$ with corresponding eigenvector $\mathbf{v} = \begin{bmatrix}-1-2i\\1\end{bmatrix}$. The conjugates of these form the second eigenvalue-eigenvector pair. The general complex solution is $\mathbf{x}(t) = c_1\begin{bmatrix}-1-2i\\1\end{bmatrix}e^{(-.4+.8i)t} + c_2\begin{bmatrix}-1+2i\\1\end{bmatrix}e^{(-.4-.8i)t}$,

where c_1 and c_2 are arbitrary complex numbers. Rewriting the first eigenfunction and taking its real and imaginary parts, we have

$$\begin{aligned}\mathbf{v}e^{(-.4+.8i)t} &= \begin{bmatrix}-1-2i\\1\end{bmatrix}e^{-.4t}(\cos .8t + i\sin .8t)\\ &= \begin{bmatrix}-\cos .8t + 2\sin .8t\\\cos .8t\end{bmatrix}e^{-.4t} + i\begin{bmatrix}-\sin .8t - 2\cos .8t\\\sin .8t\end{bmatrix}e^{-.4t}\end{aligned}$$

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The general real solution has the form $\mathbf{x}(t) = c_1 \begin{bmatrix} -\cos .8t + 2\sin .8t \\ \cos .8t \end{bmatrix} e^{-.4t} + c_2 \begin{bmatrix} -\sin .8t - 2\cos .8t \\ \sin .8t \end{bmatrix} e^{-.4t}$,

where c_1 and c_2 now are real numbers. To satisfy the initial condition $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 12 \end{bmatrix}$, we solve

$c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 12 \end{bmatrix}$ to get $c_1 = 12, c_2 = -6$. We now have

$$\begin{bmatrix} i_L(t) \\ v_C(t) \end{bmatrix} = \mathbf{x}(t) = 12 \begin{bmatrix} -\cos .8t + 2\sin .8t \\ \cos .8t \end{bmatrix} e^{-.4t} - 6 \begin{bmatrix} -\sin .8t - 2\cos .8t \\ \sin .8t \end{bmatrix} e^{-.4t} = \begin{bmatrix} 30\sin .8t \\ 12\cos .8t - 6\sin .8t \end{bmatrix} e^{-.4t}$$

5.8 - Iterative Estimates for Eigenvalues

1. The vectors in the given sequence approach an eigenvector $\mathbf{v}_1$. The last vector in the sequence, $\mathbf{x}_4 = \begin{bmatrix} 1 \\ .3326 \end{bmatrix}$, is probably the best estimate for $\mathbf{v}_1$. To compute an estimate for λ_1, examine $A\mathbf{x}_4 = \begin{bmatrix} 4.9978 \\ 1.6652 \end{bmatrix}$. This vector is approximately $\lambda_1 \mathbf{v}_1$. From the first entry in this vector, an estimate of λ_1 is 4.9978.

2. The vectors in the given sequence approach an eigenvector $\mathbf{v}_1$. The last vector in the sequence, $\mathbf{x}_4 = \begin{bmatrix} -.2520 \\ 1 \end{bmatrix}$, is probably the best estimate for $\mathbf{v}_1$. To compute an estimate for λ_1, examine $A\mathbf{x}_4 = \begin{bmatrix} -1.2536 \\ 5.0064 \end{bmatrix}$. This vector is approximately $\lambda_1 \mathbf{v}_1$. From the second entry in this vector, an estimate of λ_1 is 5.0064.

3. The vectors in the given sequence approach an eigenvector $\mathbf{v}_1$. The last vector in the sequence, $\mathbf{x}_4 = \begin{bmatrix} .5188 \\ 1 \end{bmatrix}$, is probably the best estimate for $\mathbf{v}_1$. To compute an estimate for λ_1, examine $A\mathbf{x}_4 = \begin{bmatrix} .4594 \\ .9075 \end{bmatrix}$. This vector is approximately $\lambda_1 \mathbf{v}_1$. From the second entry in this vector, an estimate of λ_1 is .9075.

4. The vectors in the given sequence approach an eigenvector $\mathbf{v}_1$. The last vector in the sequence, $\mathbf{x}_4 = \begin{bmatrix} 1 \\ .7502 \end{bmatrix}$, is probably the best estimate for $\mathbf{v}_1$. To compute an estimate for λ_1, examine $A\mathbf{x}_4 = \begin{bmatrix} -.4012 \\ -.3009 \end{bmatrix}$. This vector is approximately $\lambda_1 \mathbf{v}_1$. From the first entry in this vector, an estimate of λ_1 is $-.4012$.

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5. Since $A^5\mathbf{x} = \begin{bmatrix} 24991 \\ -31241 \end{bmatrix}$ is an estimate for an eigenvector, the vector $\mathbf{v} = -\frac{1}{31241}\begin{bmatrix} 24991 \\ -31241 \end{bmatrix} = \begin{bmatrix} -.7999 \\ 1 \end{bmatrix}$ is a vector with a 1 in its second entry that is close to an eigenvector of A. To estimate the dominant eigenvalue λ_1 of A, compute $A\mathbf{v} = \begin{bmatrix} 4.0015 \\ -5.0020 \end{bmatrix}$. From the second entry in this vector, an estimate of λ_1 is -5.0020.

6. Since $A^5\mathbf{x} = \begin{bmatrix} -2045 \\ 4093 \end{bmatrix}$ is an estimate for an eigenvector, the vector $\mathbf{v} = \frac{1}{4093}\begin{bmatrix} -2045 \\ 4093 \end{bmatrix} = \begin{bmatrix} -.4996 \\ 1 \end{bmatrix}$ is a vector with a 1 in its second entry that is close to an eigenvector of A. To estimate the dominant eigenvalue λ_1 of A, compute $A\mathbf{v} = \begin{bmatrix} -2.0008 \\ 4.0024 \end{bmatrix}$. From the second entry in this vector, an estimate of λ_1 is 4.0024.

7. $A = \begin{bmatrix} 6 & 7 \\ 8 & 5 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The data in the table below was calculated using Mathematica, which carried more digits than shown here.

k	0	1	2	3	4	5
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} .75 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ .9565 \end{bmatrix}$	$\begin{bmatrix} .9932 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ .9990 \end{bmatrix}$	$\begin{bmatrix} .9998 \\ 1 \end{bmatrix}$
$A\mathbf{x}_k$	$\begin{bmatrix} 6 \\ 8 \end{bmatrix}$	$\begin{bmatrix} 11.5 \\ 11.0 \end{bmatrix}$	$\begin{bmatrix} 12.6957 \\ 12.7826 \end{bmatrix}$	$\begin{bmatrix} 12.9592 \\ 12.9456 \end{bmatrix}$	$\begin{bmatrix} 12.9927 \\ 12.9948 \end{bmatrix}$	$\begin{bmatrix} 12.9990 \\ 12.9987 \end{bmatrix}$
μ_k	8	11.5	12.7826	12.9592	12.9948	12.9990

The actual eigenvalue is 13.

8. $A = \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The data in the table below was calculated using Mathematica, which carried more digits than shown here.

k	0	1	2	3	4	5
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} .5 \\ 1 \end{bmatrix}$	$\begin{bmatrix} .2857 \\ 1 \end{bmatrix}$	$\begin{bmatrix} .2558 \\ 1 \end{bmatrix}$	$\begin{bmatrix} .2510 \\ 1 \end{bmatrix}$	$\begin{bmatrix} .2502 \\ 1 \end{bmatrix}$
$A\mathbf{x}_k$	$\begin{bmatrix} 2 \\ 4 \end{bmatrix}$	$\begin{bmatrix} 2 \\ 7 \end{bmatrix}$	$\begin{bmatrix} 1.5714 \\ 6.1429 \end{bmatrix}$	$\begin{bmatrix} 1.5116 \\ 6.0233 \end{bmatrix}$	$\begin{bmatrix} 1.5019 \\ 6.0039 \end{bmatrix}$	$\begin{bmatrix} 1.5003 \\ 6.0006 \end{bmatrix}$
μ_k	4	7	6.1429	6.0233	6.0039	6.0006

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The actual eigenvalue is 6.

9. $A=\begin{bmatrix}8 & 0 & 12\\ 1 & -2 & 1\\ 0 & 3 & 0\end{bmatrix}$, $\mathbf{x}_0=\begin{bmatrix}1\\0\\0\end{bmatrix}$. The data in the table below was calculated using Mathematica, which carried more digits than shown here.

k	0	1	2	3	4	5	6
$\mathbf{x}_k$	$\begin{bmatrix}1\\0\\0\end{bmatrix}$	$\begin{bmatrix}1\\.125\\0\end{bmatrix}$	$\begin{bmatrix}1\\.0938\\.0469\end{bmatrix}$	$\begin{bmatrix}1\\.1004\\.0328\end{bmatrix}$	$\begin{bmatrix}1\\.0991\\.0359\end{bmatrix}$	$\begin{bmatrix}1\\.0994\\.0353\end{bmatrix}$	$\begin{bmatrix}1\\.0993\\.0354\end{bmatrix}$
$A\mathbf{x}_k$	$\begin{bmatrix}8\\1\\0\end{bmatrix}$	$\begin{bmatrix}8\\.75\\.375\end{bmatrix}$	$\begin{bmatrix}8.5625\\.8594\\.2812\end{bmatrix}$	$\begin{bmatrix}8.3942\\.8321\\.3011\end{bmatrix}$	$\begin{bmatrix}8.4304\\.8376\\.2974\end{bmatrix}$	$\begin{bmatrix}8.4233\\.8366\\.2981\end{bmatrix}$	$\begin{bmatrix}8.4246\\.8368\\.2979\end{bmatrix}$
μ_k	8	8	8.5625	8.3942	8.4304	8.4233	8.4246

Thus $\mu_5=8.4233$ and $\mu_6=8.4246$. The actual eigenvalue is $(7+\sqrt{97})/2$, or 8.42443 to five decimal places.

10. $A=\begin{bmatrix}1 & 2 & -2\\ 1 & 1 & 9\\ 0 & 1 & 9\end{bmatrix}$, $\mathbf{x}_0=\begin{bmatrix}1\\0\\0\end{bmatrix}$. The data in the table below was calculated using Mathematica, which carried more digits than shown here.

k	0	1	2	3	4	5	6
$\mathbf{x}_k$	$\begin{bmatrix}1\\0\\0\end{bmatrix}$	$\begin{bmatrix}1\\1\\0\end{bmatrix}$	$\begin{bmatrix}1\\.6667\\.3333\end{bmatrix}$	$\begin{bmatrix}.3571\\1\\.7857\end{bmatrix}$	$\begin{bmatrix}.0932\\1\\.9576\end{bmatrix}$	$\begin{bmatrix}.0183\\1\\.9904\end{bmatrix}$	$\begin{bmatrix}.0038\\1\\.9982\end{bmatrix}$
$A\mathbf{x}_k$	$\begin{bmatrix}1\\1\\0\end{bmatrix}$	$\begin{bmatrix}3\\2\\1\end{bmatrix}$	$\begin{bmatrix}1.6667\\4.6667\\3.6667\end{bmatrix}$	$\begin{bmatrix}.7857\\8.4286\\8.0714\end{bmatrix}$	$\begin{bmatrix}.1780\\9.7119\\9.6186\end{bmatrix}$	$\begin{bmatrix}.0375\\9.9319\\9.9136\end{bmatrix}$	$\begin{bmatrix}.0075\\9.9872\\9.9834\end{bmatrix}$
μ_k	1	3	4.6667	8.4286	9.7119	9.9319	9.9872

Thus $\mu_5=9.9319$ and $\mu_6=9.9872$. The actual eigenvalue is 10.

11. $A=\begin{bmatrix}5 & 2\\ 2 & 2\end{bmatrix}$, $\mathbf{x}_0=\begin{bmatrix}1\\0\end{bmatrix}$. The data in the table below was calculated using Mathematica, which carried more digits than shown here.

k	0	1	2	3	4

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$\mathbf{x}_k$	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}1\\.4\end{bmatrix}$	$\begin{bmatrix}1\\.4828\end{bmatrix}$	$\begin{bmatrix}1\\.4971\end{bmatrix}$	$\begin{bmatrix}1\\.4995\end{bmatrix}$
$A\mathbf{x}_k$	$\begin{bmatrix}5\\2\end{bmatrix}$	$\begin{bmatrix}5.8\\2.8\end{bmatrix}$	$\begin{bmatrix}5.9655\\2.9655\end{bmatrix}$	$\begin{bmatrix}5.9942\\2.9942\end{bmatrix}$	$\begin{bmatrix}5.9990\\2.9990\end{bmatrix}$
μ_k	5	5.8	5.9655	5.9942	5.9990
$R(\mathbf{x}_k)$	5	5.9655	5.9990	5.99997	5.9999993

The actual eigenvalue is 6. The bottom two columns of the table show that $R(\mathbf{x}_k)$ estimates the eigenvalue more accurately than μ_k.

12. $A=\begin{bmatrix}-3 & 2\\ 2 & 0\end{bmatrix}, \mathbf{x}_0=\begin{bmatrix}1\\0\end{bmatrix}$. The data in the table below was calculated using Mathematica, which carried more digits than shown here.

k	0	1	2	3	4
$\mathbf{x}_k$	$\begin{bmatrix}1\\0\end{bmatrix}$	$\begin{bmatrix}-1\\.6667\end{bmatrix}$	$\begin{bmatrix}1\\-.4615\end{bmatrix}$	$\begin{bmatrix}-1\\.5098\end{bmatrix}$	$\begin{bmatrix}1\\-.4976\end{bmatrix}$
$A\mathbf{x}_k$	$\begin{bmatrix}-3\\2\end{bmatrix}$	$\begin{bmatrix}4.3333\\-2.0000\end{bmatrix}$	$\begin{bmatrix}-3.9231\\2.0000\end{bmatrix}$	$\begin{bmatrix}4.0196\\-2.0000\end{bmatrix}$	$\begin{bmatrix}-3.9951\\2.0000\end{bmatrix}$
μ_k	–3	–4.3333	–3.9231	–4.0196	–3.9951
$R(\mathbf{x}_k)$	–3	–3.9231	–3.9951	–3.9997	–3.99998

The actual eigenvalue is -4. The bottom two columns of the table show that $R(\mathbf{x}_k)$ estimates the eigenvalue more accurately than μ_k.

13. If the eigenvalues close to 4 and -4 have different absolute values, then one of these is a strictly dominant eigenvalue, so the power method will work. But the power method depends on powers of the quotients λ_2/λ_1 and λ_3/λ_1 going to zero. If $|\lambda_2/\lambda_1|$ is close to 1, its powers will go to zero slowly, and the power method will converge slowly.

14. If the eigenvalues close to 4 and -4 have the same absolute value, then neither of these is a strictly dominant eigenvalue, so the power method will not work. However, the inverse power method may still be used. If the initial estimate is chosen near the eigenvalue close to 4, then the inverse power method should produce a sequence that estimates the eigenvalue close to 4.

15. Suppose $A\mathbf{x}=\lambda\mathbf{x}$, with $\mathbf{x}\neq 0$. For any α, $A\mathbf{x}-\alpha I\mathbf{x}=(\lambda-\alpha)\mathbf{x}$. If α is *not* an eigenvalue of A, then $A-\alpha I$ is invertible and $\lambda-\alpha$ is not 0; hence $\mathbf{x}=(A-\alpha I)^{-1}(\lambda-\alpha)\mathbf{x}$ and $(\lambda-\alpha)^{-1}\mathbf{x}=(A-\alpha I)^{-1}\mathbf{x}$

This last equation shows that $\mathbf{x}$ is an eigenvector of $(A-\alpha I)^{-1}$ corresponding to the eigenvalue $(\lambda-\alpha)^{-1}$.

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16. Suppose that μ is an eigenvalue of $(A-\alpha I)^{-1}$ with corresponding eigenvector **x**. Since $(A-\alpha I)^{-1}\mathbf{x}=\mu\mathbf{x}$, $\mathbf{x}=(A-\alpha I)(\mu\mathbf{x})=A(\mu\mathbf{x})-(\alpha I)(\mu\mathbf{x})=\mu(A\mathbf{x})-\alpha\mu\mathbf{x}$, solving this equation for A**x**, we find that $A\mathbf{x}=\left(\frac{1}{\mu}\right)(\alpha\mu\mathbf{x}+\mathbf{x})=\left(\alpha+\frac{1}{\mu}\right)\mathbf{x}$. Thus $\lambda=\alpha+(1/\mu)$ is an eigenvalue of A with corresponding eigenvector **x**.

17. $A=\begin{bmatrix}10 & -8 & -4\\ -8 & 13 & 4\\ -4 & 5 & 4\end{bmatrix}, \mathbf{x}_0=\begin{bmatrix}1\\0\\0\end{bmatrix}, \alpha=3.3.$ The data in the table below was calculated using Mathematica, which carried more digits than shown here.

k	0	1	2
$\mathbf{x}_k$	$\begin{bmatrix}1\\0\\0\end{bmatrix}$	$\begin{bmatrix}1\\.7873\\.0908\end{bmatrix}$	$\begin{bmatrix}1\\.7870\\.0957\end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix}26.0552\\20.5128\\2.3669\end{bmatrix}$	$\begin{bmatrix}47.1975\\37.1436\\4.5187\end{bmatrix}$	$\begin{bmatrix}47.1233\\37.0866\\4.5083\end{bmatrix}$
μ_k	26.0552	47.1975	47.1233
ν_k	3.3384	3.32119	3.3212209

Thus an estimate for the eigenvalue to four decimal places is 3.3212. The actual eigenvalue is $(25-\sqrt{337})/2$, or 3.3212201 to seven decimal places.

18. $A=\begin{bmatrix}8 & 0 & 12\\ 1 & -2 & 1\\ 0 & 3 & 0\end{bmatrix}, \mathbf{x}_0=\begin{bmatrix}1\\0\\0\end{bmatrix}, \alpha=-1.4.$ The data in the table below was calculated using Mathematica, which carried more digits than shown here.

k	0	1	2	3	4
$\mathbf{x}_k$	$\begin{bmatrix}1\\0\\0\end{bmatrix}$	$\begin{bmatrix}1\\.3646\\-.7813\end{bmatrix}$	$\begin{bmatrix}1\\.3734\\-.7854\end{bmatrix}$	$\begin{bmatrix}1\\.3729\\-.7854\end{bmatrix}$	$\begin{bmatrix}1\\.3729\\-.7854\end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix}40\\14.5833\\-31.25\end{bmatrix}$	$\begin{bmatrix}-38.125\\-14.2361\\29.9479\end{bmatrix}$	$\begin{bmatrix}-41.1134\\-15.3300\\32.2888\end{bmatrix}$	$\begin{bmatrix}-40.9243\\-15.2608\\32.1407\end{bmatrix}$	$\begin{bmatrix}-40.9358\\-15.2650\\32.1497\end{bmatrix}$
μ_k	40	−38.125	−41.1134	−40.9243	−40.9358
ν_k	−1.375	−1.42623	−1.42432	−1.42444	−1.42443

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Thus an estimate for the eigenvalue to four decimal places is -1.4244. The actual eigenvalue is $(7-\sqrt{97})/2$, or -1.424429 to six decimal places.

19. $A = \begin{bmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{bmatrix}, \mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$

(a) The data in the table below was calculated using Mathematica (with $\alpha = 0$), which carried more digits than shown here.

k	0	1	2	3
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ .7 \\ .8 \\ .7 \end{bmatrix}$	$\begin{bmatrix} .988679 \\ .709434 \\ 1 \\ .932075 \end{bmatrix}$	$\begin{bmatrix} .961467 \\ .691491 \\ 1 \\ .942201 \end{bmatrix}$
$A\mathbf{x}_k$	$\begin{bmatrix} 10 \\ 7 \\ 8 \\ 7 \end{bmatrix}$	$\begin{bmatrix} 26.2 \\ 18.8 \\ 26.5 \\ 24.7 \end{bmatrix}$	$\begin{bmatrix} 29.3774 \\ 21.1283 \\ 30.5547 \\ 28.7887 \end{bmatrix}$	$\begin{bmatrix} 29.0505 \\ 20.8987 \\ 30.3205 \\ 28.6097 \end{bmatrix}$
μ_k	10	26.5	30.5547	30.3205

k	4	5	6	7
$\mathbf{x}_k$	$\begin{bmatrix} .958115 \\ .689261 \\ 1 \\ .943578 \end{bmatrix}$	$\begin{bmatrix} .957691 \\ .688978 \\ 1 \\ .943755 \end{bmatrix}$	$\begin{bmatrix} .957637 \\ .688942 \\ 1 \\ .943778 \end{bmatrix}$	$\begin{bmatrix} .957630 \\ .688938 \\ 1 \\ .943781 \end{bmatrix}$
$A\mathbf{x}_k$	$\begin{bmatrix} 29.0110 \\ 20.8710 \\ 30.2927 \\ 28.5889 \end{bmatrix}$	$\begin{bmatrix} 29.0060 \\ 20.8675 \\ 30.2892 \\ 28.5863 \end{bmatrix}$	$\begin{bmatrix} 29.0054 \\ 20.8671 \\ 30.2887 \\ 28.5859 \end{bmatrix}$	$\begin{bmatrix} 29.0053 \\ 20.8670 \\ 30.2887 \\ 28.5859 \end{bmatrix}$
μ_k	30.2927	30.2892	30.2887	30.2887

Thus an estimate for the eigenvalue to four decimal places is 30.2887. The actual eigenvalue is 30.2886853 to seven decimal places. An estimate for the corresponding eigenvector is

$$\begin{bmatrix} .957630 \\ .688938 \\ 1 \\ .943781 \end{bmatrix}.$$

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(b) The data in the table below was calculated using Mathematica (with $\alpha = 0$), which carried more digits than shown here.

k	0	1	2	3	4
$\mathbf{x}_k$	$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$	$\begin{bmatrix}-.609756\\1\\-.243902\\.146341\end{bmatrix}$	$\begin{bmatrix}-.604007\\1\\-.251051\\.148899\end{bmatrix}$	$\begin{bmatrix}-.603973\\1\\-.251134\\.148953\end{bmatrix}$	$\begin{bmatrix}-.603972\\1\\-.251135\\.148953\end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix}25\\-41\\10\\-6\end{bmatrix}$	$\begin{bmatrix}-59.5610\\98.6098\\-24.7561\\14.6829\end{bmatrix}$	$\begin{bmatrix}-59.5041\\98.5211\\-24.7420\\14.6750\end{bmatrix}$	$\begin{bmatrix}-59.5044\\98.5217\\-24.7423\\14.6751\end{bmatrix}$	$\begin{bmatrix}-59.5044\\98.5217\\-24.7423\\14.6751\end{bmatrix}$
μ_k	–41	98.6098	98.5211	98.5217	98.5217
ν_k	–.0243902	.0101410	.0101501	.0101500	.0101500

Thus an estimate for the eigenvalue to five decimal places is .01015. The actual eigenvalue is .01015005 to eight decimal places. An estimate for the corresponding eigenvector is

$$\begin{bmatrix}-.603972\\1\\-.251135\\.148953\end{bmatrix}.$$

20. $A = \begin{bmatrix}1 & 2 & 3 & 2\\2 & 12 & 13 & 11\\-2 & 3 & 0 & 2\\4 & 5 & 7 & 2\end{bmatrix}$, $\mathbf{x}_0 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$.

(a) The data in the table below was calculated using Mathematica, which carried more digits than shown here.

k	0	1	2	3	4
$\mathbf{x}_k$	$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$	$\begin{bmatrix}.25\\.5\\-.5\\1\end{bmatrix}$	$\begin{bmatrix}.159091\\1\\.272727\\.181818\end{bmatrix}$	$\begin{bmatrix}.187023\\1\\.170483\\.442748\end{bmatrix}$	$\begin{bmatrix}.184166\\1\\.180439\\.402197\end{bmatrix}$
$A\mathbf{x}_k$	$\begin{bmatrix}1\\2\\-2\\4\end{bmatrix}$	$\begin{bmatrix}1.75\\11\\3\\2\end{bmatrix}$	$\begin{bmatrix}3.34091\\17.8636\\3.04545\\7.90909\end{bmatrix}$	$\begin{bmatrix}3.58397\\19.4606\\3.51145\\7.82697\end{bmatrix}$	$\begin{bmatrix}3.52988\\19.1382\\3.43606\\7.80413\end{bmatrix}$
μ_k	4	11	17.8636	19.4606	19.1382

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k	5	6	7	8	9
$\mathbf{x}_k$	$\begin{bmatrix} .184441 \\ 1 \\ .179539 \\ .407778 \end{bmatrix}$	$\begin{bmatrix} .184414 \\ 1 \\ .179622 \\ .407021 \end{bmatrix}$	$\begin{bmatrix} .184417 \\ 1 \\ .179615 \\ .407121 \end{bmatrix}$	$\begin{bmatrix} .184416 \\ 1 \\ .179615 \\ .407108 \end{bmatrix}$	$\begin{bmatrix} .184416 \\ 1 \\ .179615 \\ .407110 \end{bmatrix}$
$A\mathbf{x}_k$	$\begin{bmatrix} 3.53861 \\ 19.1884 \\ 3.44667 \\ 7.81010 \end{bmatrix}$	$\begin{bmatrix} 3.53732 \\ 19.1811 \\ 3.44521 \\ 7.80905 \end{bmatrix}$	$\begin{bmatrix} 3.53750 \\ 19.1822 \\ 3.44541 \\ 7.80921 \end{bmatrix}$	$\begin{bmatrix} 3.53748 \\ 19.1820 \\ 3.44538 \\ 7.80919 \end{bmatrix}$	$\begin{bmatrix} 3.53748 \\ 19.1811 \\ 3.44539 \\ 7.80919 \end{bmatrix}$
μ_k	19.1884	19.1811	19.1822	19.1820	19.1820

Thus an estimate for the eigenvalue to four decimal places is 19.1820. The actual eigenvalue is 19.1820368 to seven decimal places. An estimate for the corresponding eigenvector is

$$\begin{bmatrix} .184416 \\ 1 \\ .179615 \\ .407110 \end{bmatrix}.$$

(b) The data in the table below was calculated using Mathematica, which carried more digits than shown here.

k	0	1	2
$\mathbf{x}_k$	$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ .226087 \\ -.921739 \\ .660870 \end{bmatrix}$	$\begin{bmatrix} 1 \\ .222577 \\ -.917970 \\ .660496 \end{bmatrix}$
$\mathbf{y}_k$	$\begin{bmatrix} 115 \\ 26 \\ -106 \\ 76 \end{bmatrix}$	$\begin{bmatrix} 81.7304 \\ 18.1913 \\ -75.0261 \\ 53.9826 \end{bmatrix}$	$\begin{bmatrix} 81.9314 \\ 18.2387 \\ -75.2125 \\ 54.1143 \end{bmatrix}$
μ_k	115	81.7304	81.9314
ν_k	.00869565	.0122353	.0122053

Thus an estimate for the eigenvalue to four decimal places is .0122. The actual eigenvalue is .01220556 to eight decimal places. An estimate for the corresponding eigenvector is

$$\begin{bmatrix} 1 \\ .222577 \\ -.917970 \\ .660496 \end{bmatrix}.$$

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21. **a**. $A=\begin{bmatrix}.8 & 0\\ 0 & .2\end{bmatrix}, \mathbf{x}=\begin{bmatrix}.5\\ .5\end{bmatrix}$. Here is the sequence $A^k\mathbf{x}$ for $k=1,\dots 5$:

$\begin{bmatrix}.4\\ .1\end{bmatrix},\begin{bmatrix}.32\\ .02\end{bmatrix},\begin{bmatrix}.256\\ .004\end{bmatrix},\begin{bmatrix}.2048\\ .0008\end{bmatrix},\begin{bmatrix}.16384\\ .00016\end{bmatrix}$. Notice that $A^5\mathbf{x}$ is approximately $.8(A^4\mathbf{x})$.

Conclusion: If the eigenvalues of A are all less than 1 in magnitude, and if $\mathbf{x}\neq 0$, then $A^k\mathbf{x}$ is approximately an eigenvector for large k.

b. $A=\begin{bmatrix}1 & 0\\ 0 & .8\end{bmatrix}, \mathbf{x}=\begin{bmatrix}.5\\ .5\end{bmatrix}$. Here is the sequence $A^k\mathbf{x}$ for $k=1,\dots 5$:

$\begin{bmatrix}.5\\ .4\end{bmatrix},\begin{bmatrix}.5\\ .32\end{bmatrix},\begin{bmatrix}.5\\ .256\end{bmatrix},\begin{bmatrix}.5\\ .2048\end{bmatrix},\begin{bmatrix}.5\\ .16384\end{bmatrix}$. Notice that $A^k\mathbf{x}$ seems to be converging to $\begin{bmatrix}.5\\ 0\end{bmatrix}$.

Conclusion: If the strictly dominant eigenvalue of A is 1, and if $\mathbf{x}$ has a component in the direction of the corresponding eigenvector, then $\{A^k\mathbf{x}\}$ will converge to a multiple of that eigenvector.

c. $A=\begin{bmatrix}8 & 0\\ 0 & 2\end{bmatrix}, \mathbf{x}=\begin{bmatrix}.5\\ .5\end{bmatrix}$. Here is the sequence $A^k\mathbf{x}$ for $k=1,\dots 5$:

$\begin{bmatrix}4\\ 1\end{bmatrix},\begin{bmatrix}32\\ 2\end{bmatrix},\begin{bmatrix}256\\ 4\end{bmatrix},\begin{bmatrix}2048\\ 8\end{bmatrix},\begin{bmatrix}16384\\ 16\end{bmatrix}$. Notice that the distance of $A^k\mathbf{x}$ from either eigenvector of A is increasing rapidly as k increases.

Conclusion: If the eigenvalues of A are all greater than 1 in magnitude, and if $\mathbf{x}$ is not an eigenvector, then the distance from $A^k\mathbf{x}$ to the nearest eigenvector will *increase* as $k\to\infty$.

5.9 - Applications to Markov Chains

Notes: This section builds on the population movement example in Section 1.10. The migration matrix is examined again in Section 5.2, where an eigenvector decomposition shows explicitly why the sequence of state vectors $\mathbf{x}_k$ tends to a steady state vector. The discussion in Section 5.2 does not depend on prior knowledge of this section.

1. **a**. Let N stand for "News" and M stand for "Music." Then the listeners' behavior is given by the table

From:		
N	M	To:
.7	.6	N
.3	.4	M

so the stochastic matrix is $P=\begin{bmatrix}.7 & .6\\ .3 & .4\end{bmatrix}$.

b. Since 100% of the listeners are listening to news at 8:15, the initial state vector is $\mathbf{x}_0=\begin{bmatrix}1\\ 0\end{bmatrix}$.

c. There are two breaks between 8:15 and 9:25, so we calculate $\mathbf{x}_2$:

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$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} .7 & .6 \\ .3 & .4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .7 \\ .3 \end{bmatrix}$$

$$\mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} .7 & .6 \\ .3 & .4 \end{bmatrix}\begin{bmatrix} .7 \\ .3 \end{bmatrix} = \begin{bmatrix} .67 \\ .33 \end{bmatrix}$$

Thus 33% of the listeners are listening to news at 9:25.

2. **a**. Let the foods be labelled "1," "2," and "3." Then the animals' behavior is given by the table

From:			
1	2	3	To:
.5	.25	.25	1
.25	.5	.25	2
.25	.25	.5	3

so the stochastic matrix is $P = \begin{bmatrix} .5 & .25 & .25 \\ .25 & .5 & .25 \\ .25 & .25 & .5 \end{bmatrix}$.

b. There are two trials after the initial trial, so we calculate $\mathbf{x}_2$. The initial state vector is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} .5 & .25 & .25 \\ .25 & .5 & .25 \\ .25 & .25 & .5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} .5 \\ .25 \\ .25 \end{bmatrix}$$

$$\mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} .5 & .25 & .25 \\ .25 & .5 & .25 \\ .25 & .25 & .5 \end{bmatrix}\begin{bmatrix} .5 \\ .25 \\ .25 \end{bmatrix} = \begin{bmatrix} .375 \\ .3125 \\ .3125 \end{bmatrix}$$

Thus the probability that the animal will choose food #2 is .3125.

3. **a**. Let H stand for "Healthy" and I stand for "Ill." Then the students' conditions are given by the table

From:		
H	I	To:
.95	.45	H
.05	.55	I

so the stochastic matrix is $P = \begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix}$.

b. Since 20% of the students are ill on Monday, the initial state vector is $\mathbf{x}_0 = \begin{bmatrix} .8 \\ .2 \end{bmatrix}$. For Tuesday's percentages, we calculate $\mathbf{x}_1$; for Wednesday's percentages, we calculate $\mathbf{x}_2$:

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix}\begin{bmatrix} .8 \\ .2 \end{bmatrix} = \begin{bmatrix} .85 \\ .15 \end{bmatrix}$$

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$$\mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix}\begin{bmatrix} .85 \\ .15 \end{bmatrix} = \begin{bmatrix} .875 \\ .125 \end{bmatrix}$$

Thus 15% of the students are ill on Tuesday, and 12.5% are ill on Wednesday.

c. Since the student is well today, the initial state vector is $\mathbf{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We calculate $\mathbf{x}_2$:

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .95 \\ .05 \end{bmatrix}$$

$$\mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix}\begin{bmatrix} .95 \\ .05 \end{bmatrix} = \begin{bmatrix} .925 \\ .075 \end{bmatrix}$$

Thus the probability that the student is well two days from now is .925.

4. **a**. Let G stand for good weather, I for indifferent weather, and B for bad weather. Then the change in the weather is given by the table

From:			
G	I	B	To:
.6	.4	.4	G
.3	.3	.5	I
.1	.3	.1	B

so the stochastic matrix is $P = \begin{bmatrix} .6 & .4 & .4 \\ .3 & .3 & .5 \\ .1 & .3 & .1 \end{bmatrix}$.

b. The initial state vector is $\begin{bmatrix} .5 \\ .5 \\ 0 \end{bmatrix}$. We calculate $\mathbf{x}_1$:

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} .6 & .4 & .4 \\ .3 & .3 & .5 \\ .1 & .3 & .1 \end{bmatrix}\begin{bmatrix} .5 \\ .5 \\ 0 \end{bmatrix} = \begin{bmatrix} .5 \\ .3 \\ .2 \end{bmatrix}$$

Thus the chance of bad weather tomorrow is 20%.

c. The initial state vector is $\mathbf{x}_0 = \begin{bmatrix} 0 \\ .4 \\ .6 \end{bmatrix}$. We calculate $\mathbf{x}_2$:

$$\mathbf{x}_1 = P\mathbf{x}_0 = \begin{bmatrix} .6 & .4 & .4 \\ .3 & .3 & .5 \\ .1 & .3 & .1 \end{bmatrix}\begin{bmatrix} 0 \\ .4 \\ .6 \end{bmatrix} = \begin{bmatrix} .4 \\ .42 \\ .18 \end{bmatrix}$$

$$\mathbf{x}_2 = P\mathbf{x}_1 = \begin{bmatrix} .6 & .4 & .4 \\ .3 & .3 & .5 \\ .1 & .3 & .1 \end{bmatrix}\begin{bmatrix} .4 \\ .42 \\ .18 \end{bmatrix} = \begin{bmatrix} .48 \\ .336 \\ .184 \end{bmatrix}$$

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Thus the chance of good weather on Wednesday is 48%.

5. We solve $P\mathbf{x} = \mathbf{x}$ by rewriting the equation as $(P - I)\mathbf{x} = \mathbf{0}$, where $P - I = \begin{bmatrix} -.9 & .6 \\ .9 & -.6 \end{bmatrix}$. Row reducing the augmented matrix for the homogeneous system $(P - I)\mathbf{x} = \mathbf{0}$ gives

$$\begin{bmatrix} -.9 & .6 & 0 \\ .9 & -.6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2/3 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ Thus $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$, and one solution is $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Since the entries in $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ sum to 5, multiply by 1/5 to obtain the steady-state vector $\mathbf{q} = \begin{bmatrix} 2/5 \\ 3/5 \end{bmatrix} = \begin{bmatrix} .4 \\ .6 \end{bmatrix}$.

6. We solve $P\mathbf{x} = \mathbf{x}$ by rewriting the equation as $(P - I)\mathbf{x} = \mathbf{0}$, where $P - I = \begin{bmatrix} -.2 & .5 \\ .2 & -.5 \end{bmatrix}$. Row reducing the augmented matrix for the homogeneous system $(P - I)\mathbf{x} = \mathbf{0}$ gives

$$\begin{bmatrix} -.2 & .5 & 0 \\ .2 & -.5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -5/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ Thus $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 5/2 \\ 1 \end{bmatrix}$, and one solution is $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$. Since the entries in $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ sum to 7, multiply by 1/7 to obtain the steady-state vector $\mathbf{q} = \begin{bmatrix} 5/7 \\ 2/7 \end{bmatrix} \approx \begin{bmatrix} .714 \\ .286 \end{bmatrix}$.

7. We solve $P\mathbf{x} = \mathbf{x}$ by rewriting the equation as $(P - I)\mathbf{x} = \mathbf{0}$, where $P - I = \begin{bmatrix} -.3 & .1 & .1 \\ .2 & -.2 & .2 \\ .1 & .1 & -.3 \end{bmatrix}$. Row reducing the augmented matrix for the homogeneous system $(P - I)\mathbf{x} = \mathbf{0}$ gives

$$\begin{bmatrix} -.3 & .1 & .1 & 0 \\ .2 & -.2 & .2 & 0 \\ .1 & .1 & -.3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$ Thus $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, and one solution is $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. Since the entries in $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ sum to 4, multiply by 1/4 to obtain the steady-state vector $\mathbf{q} = \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix} = \begin{bmatrix} .25 \\ .5 \\ .25 \end{bmatrix}$.

8. We solve $P\mathbf{x} = \mathbf{x}$ by rewriting the equation as $(P - I)\mathbf{x} = \mathbf{0}$, where $P - I = \begin{bmatrix} -.3 & .2 & .2 \\ 0 & -.8 & .4 \\ .3 & .6 & -.6 \end{bmatrix}$. Row reducing the augmented matrix for the homogeneous system $(P - I)\mathbf{x} = \mathbf{0}$ gives

$$\begin{bmatrix} -.3 & .2 & .2 & 0 \\ 0 & -.8 & .4 & 0 \\ .3 & .6 & -.6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$ Thus $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix}$, and one solution is $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

Since the entries in $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ sum to 5, multiply by 1/5 to obtain the steady-state vector $\mathbf{q} = \begin{bmatrix} 2/5 \\ 1/5 \\ 2/5 \end{bmatrix} = \begin{bmatrix} .4 \\ .2 \\ .4 \end{bmatrix}$.

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9. Since $P^2 = \begin{bmatrix} .84 & .2 \\ .16 & .8 \end{bmatrix}$ has all positive entries, P is a regular stochastic matrix.

10. Since $P^k = \begin{bmatrix} 1 & 1-.8^k \\ 0 & .8^k \end{bmatrix}$ will have a zero as its (2,1) entry for all k, so P is not a regular stochastic matrix.

11. From Exercise 1, $P = \begin{bmatrix} .7 & .6 \\ .3 & .4 \end{bmatrix}$, so $P - I = \begin{bmatrix} -.3 & .6 \\ .3 & -.6 \end{bmatrix}$. Solving $(P - I)\mathbf{x} = \mathbf{0}$ by row reducing the augmented matrix gives $\begin{bmatrix} -.3 & .6 & 0 \\ .3 & -.6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and one solution is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Since the entries in $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ sum to 3, multiply by 1/3 to obtain the steady-state vector $\mathbf{q} = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} \approx \begin{bmatrix} .667 \\ .333 \end{bmatrix}$.

12. From Exercise 2, $P = \begin{bmatrix} .5 & .25 & .25 \\ .25 & .5 & .25 \\ .25 & .25 & .5 \end{bmatrix}$, so $P - I = \begin{bmatrix} -.5 & .25 & .25 \\ .25 & -.5 & .25 \\ .25 & .25 & -.5 \end{bmatrix}$. Solving $(P - I)\mathbf{x} = \mathbf{0}$ by row reducing the augmented matrix gives $\begin{bmatrix} -.5 & .25 & .25 & 0 \\ .25 & -.5 & .25 & 0 \\ .25 & .25 & -.5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Thus $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and one solution is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Since the entries in $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ sum to 3, multiply by 1/3 to obtain the steady-state vector $\mathbf{q} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \end{bmatrix} \approx \begin{bmatrix} .333 \\ .333 \\ .333 \end{bmatrix}$. Thus in the long run each food will be preferred equally.

13. **a**. From Exercise 3, $P = \begin{bmatrix} .95 & .45 \\ .05 & .55 \end{bmatrix}$, so $P - I = \begin{bmatrix} -.05 & .45 \\ .05 & -.45 \end{bmatrix}$. Solving $(P - I)\mathbf{x} = \mathbf{0}$ by row reducing the augmented matrix gives $\begin{bmatrix} -.05 & .45 & 0 \\ .05 & -.45 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -9 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 9 \\ 1 \end{bmatrix}$, and one solution is $\begin{bmatrix} 9 \\ 1 \end{bmatrix}$. Since the entries in $\begin{bmatrix} 9 \\ 1 \end{bmatrix}$ sum to 10, multiply by 1/10 to obtain the steady-state vector $\mathbf{q} = \begin{bmatrix} 9/10 \\ 1/10 \end{bmatrix} = \begin{bmatrix} .9 \\ .1 \end{bmatrix}$.

b. After many days, a specific student is ill with probability .1, and it does not matter whether that student is ill today or not.

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14. From Exercise 4, $P=\begin{bmatrix} .6 & .4 & .4 \\ .3 & .3 & .5 \\ .1 & .3 & .1 \end{bmatrix}$, so $P-I=\begin{bmatrix} -.4 & .4 & .4 \\ .3 & -.7 & .5 \\ .1 & .3 & -.9 \end{bmatrix}$. Solving $(P-I)\mathbf{x}=\mathbf{0}$ by row reducing the augmented matrix gives $\begin{bmatrix} -.4 & .4 & .4 & 0 \\ .3 & -.7 & .5 & 0 \\ .1 & .3 & -.9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Thus $\mathbf{x}=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=x_3\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, and one solution is $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. Since the entries in $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ sum to 6, multiply by 1/6 to obtain the steady-state vector $\mathbf{q}=\begin{bmatrix} 1/2 \\ 1/3 \\ 1/6 \end{bmatrix} \approx \begin{bmatrix} .5 \\ .333 \\ .167 \end{bmatrix}$. Thus in the long run the chance that a day has good weather is 50%.

15. True. See the paragraph before Example 4.

16. False. Only the eigenvector corresponding to the largest eigenvalue.

17. True. See the proof of Theorem 10.

18. False. See the paragraph before Example 4.

19. True. It can be one of the eigenvalues that is smaller than 1..

20. False. The identity is stochastic but not regular.

21. No. The vector **q** is not a probability vector since its entries do not add to 1..

22. No. The vector **q** is not a probability vector since its entries do not add to 1.

23. No. The vector $A\mathbf{q}$ does not equal **q.**

24. No. The vector $A\mathbf{q}$ does not equal **q.**

25. $\begin{bmatrix} .70 & .15 \\ .30 & .85 \end{bmatrix}-\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} -.30 & .15 \\ .30 & -.15 \end{bmatrix}$, so the steady state vector is given by $\begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$. The second entry corresponds to the number of Android users, hence about 67% of cell phone users will have an Android operating system.

26. Let $P=\begin{bmatrix} .90 & .01 & .09 \\ .01 & .90 & .01 \\ .09 & .09 & .90 \end{bmatrix}$, so $P-I=\begin{bmatrix} -.10 & .01 & .09 \\ .01 & -.10 & .01 \\ .09 & .09 & -.1 \end{bmatrix}$. Solving $(P-I)\mathbf{x}=\mathbf{0}$ by row reducing the augmented matrix gives $\begin{bmatrix} -.10 & .01 & .09 & 0 \\ .01 & -.10 & .01 & 0 \\ .09 & .09 & -.1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -.919192 & 0 \\ 0 & 1 & -.191919 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

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Thus $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} .919192 \\ .191919 \\ 1 \end{bmatrix}$, and one solution is $\begin{bmatrix} .919192 \\ .191919 \\ 1 \end{bmatrix}$. Since the entries in $\begin{bmatrix} .919192 \\ .191919 \\ 1 \end{bmatrix}$ sum to 2.111111, multiply by 1/2.111111 to obtain the steady-state vector $\mathbf{q} = \begin{bmatrix} .435407 \\ .090909 \\ .473684 \end{bmatrix}$. Thus on a typical day, about (.090909)(2000) = 182 cars will be rented or available from the downtown location.

27. **a**. The entries in each column of P sum to 1. Each column in the matrix $P - I$ has the same entries as in P except one of the entries is decreased by 1. Thus the entries in each column of $P - I$ sum to 0, and adding all of the other rows of $P - I$ to its bottom row produces a row of zeros.

b. By part a., the bottom row of $P - I$ is the negative of the sum of the other rows, so the rows of $P - I$ are linearly dependent.

c. By part b. and the Spanning Set Theorem, the bottom row of $P - I$ can be removed and the remaining $(n - 1)$ rows will still span the row space of $P - I$. Thus the dimension of the row space of $P - I$ is less than n. Alternatively, let A be the matrix obtained from $P - I$ by adding to the bottom row all the other rows. These row operations did not change the row space, so the row space of $P - I$ is spanned by the nonzero rows of A. By part a., the bottom row of A is a zero row, so the row space of $P - I$ is spanned by the first $(n - 1)$ rows of A.

d. By part c., the rank of $P - I$ is less than n, so the Rank Theorem may be used to show that $\dim \text{Nul}(P - I) = n - \text{rank}(P - I) > 0$. Alternatively the Invertible Martix Theorem may be used since $P - I$ is a square matrix.

28. If $\alpha = \beta = 0$ then $P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Notice that $P\mathbf{x} = \mathbf{x}$ for any vector $\mathbf{x}$ in $\mathbb{R}^2$, and that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are two linearly independent steady-state vectors in this case.

If $\alpha \neq 0$ or $\beta \neq 0$, we solve $(P - I)\mathbf{x} = \mathbf{0}$ where $P - I = \begin{bmatrix} -\alpha & \beta \\ \alpha & -\beta \end{bmatrix}$. Row reducing the augmented matrix gives $\begin{bmatrix} -\alpha & \beta & 0 \\ \alpha & -\beta & 0 \end{bmatrix} \sim \begin{bmatrix} \alpha & -\beta & 0 \\ 0 & 0 & 0 \end{bmatrix}$

So $\alpha x_1 = \beta x_2$, and one possible solution is to let $x_1 = \beta$, $x_2 = \alpha$. Thus $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$. Since the entries in $\begin{bmatrix} \beta \\ \alpha \end{bmatrix}$ sum to $\alpha + \beta$, multiply by $1/(\alpha + \beta)$ to obtain the steady-state vector $\mathbf{q} = \dfrac{1}{\alpha + \beta} \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$.

29. **a**. The product $S\mathbf{x}$ equals the sum of the entries in $\mathbf{x}$. Thus $\mathbf{x}$ is a probability vector if and only if its entries are nonnegative and $S\mathbf{x} = 1$.

b. Let $P = \begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \dots & \mathbf{p}_n \end{bmatrix}$, where $\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n$ are probability vectors. By part a.,

$$SP = \begin{bmatrix} S\mathbf{p}_1 & S\mathbf{p}_2 & \dots & S\mathbf{p}_n \end{bmatrix} = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix} = S$$

c. By part b., $S(P\mathbf{x}) = (SP)\mathbf{x} = S\mathbf{x} = 1$. The entries in $P\mathbf{x}$ are nonnegative since P and $\mathbf{x}$ have only nonnegative entries. By part a., the condition $S(P\mathbf{x}) = 1$ shows that $P\mathbf{x}$ is a probability vector.

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30. Let $P=[\mathbf{p}_1 \quad \mathbf{p}_2 \quad \dots \quad \mathbf{p}_n]$, so $P^2=PP=[P\mathbf{p}_1 \quad P\mathbf{p}_2 \quad \dots \quad P\mathbf{p}_n]$. By Exercise 19c., the columns of P^2 are probability vectors, so P^2 is a stochastic matrix. Alternatively, $SP = S$ by Exercise 19b., since P is a stochastic matrix. Right multiplication by P gives $SP^2 = SP$, so $SP = S$ implies that $SP^2 = S$. Since the entries in P are nonnegative, so are the entries in P^2, and P^2 is stochastic matrix.

31. **a**. To four decimal places $P^2=\begin{bmatrix} .2779 & .2780 & .2803 & .2941 \\ .3368 & .3355 & .3357 & .3335 \\ .1847 & .1861 & .1833 & .1697 \\ .2005 & .2004 & .2007 & .2027 \end{bmatrix}$, $P^3=\begin{bmatrix} .2817 & .2817 & .2817 & .2814 \\ .3356 & .3356 & .3355 & .3352 \\ .1817 & .1817 & .1819 & .1825 \\ .2010 & .2010 & .2010 & .2009 \end{bmatrix}$,

$P^4=P^5=\begin{bmatrix} .2816 & .2816 & .2816 & .2816 \\ .3355 & .3355 & .3355 & .3355 \\ .1819 & .1819 & .1819 & .1819 \\ .2009 & .2009 & .2009 & .2009 \end{bmatrix}$. The columns of P^k are converging to a common vector as k increases. The steady state vector $\mathbf{q}$ for P is $\mathbf{q}=\begin{bmatrix} .2816 \\ .3355 \\ .1819 \\ .2009 \end{bmatrix}$, which is the vector to which the columns of P^k are converging.

b. To four decimal places,

$$Q^{10}=\begin{bmatrix} .8222 & .4044 & .5385 \\ .0324 & .3966 & .1666 \\ .1453 & .1990 & .2949 \end{bmatrix}, Q^{20}=\begin{bmatrix} .7674 & .6000 & .6690 \\ .0637 & .2036 & .1326 \\ .1688 & .1964 & .1984 \end{bmatrix},$$

$$Q^{30}=\begin{bmatrix} .7477 & .6815 & .7105 \\ .0783 & .1329 & .1074 \\ .1740 & .1856 & .1821 \end{bmatrix}, Q^{40}=\begin{bmatrix} .7401 & .7140 & .7257 \\ .0843 & .1057 & .0960 \\ .1756 & .1802 & .1783 \end{bmatrix},$$

$$Q^{50}=\begin{bmatrix} .7372 & .7269 & .7315 \\ .0867 & .0951 & .0913 \\ .1761 & .1780 & .1772 \end{bmatrix}, Q^{60}=\begin{bmatrix} .7360 & .7320 & .7338 \\ .0876 & .0909 & .0894 \\ .1763 & .1771 & .1767 \end{bmatrix},$$

$$Q^{70}=\begin{bmatrix} .7356 & .7340 & .7347 \\ .0880 & .0893 & .0887 \\ .1764 & .1767 & .1766 \end{bmatrix}, Q^{80}=\begin{bmatrix} .7354 & .7348 & .7351 \\ .0881 & .0887 & .0884 \\ .1764 & .1766 & .1765 \end{bmatrix},$$

$$Q^{116}=Q^{117}=\begin{bmatrix} .7353 & .7353 & .7353 \\ .0882 & .0882 & .0882 \\ .1765 & .1765 & .1765 \end{bmatrix}$$

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The steady state vector $\mathbf{q}$ for Q is $\mathbf{q} = \begin{bmatrix} .7353 \\ .0882 \\ .1765 \end{bmatrix}$ Conjecture: the columns of P^k, where P is a regular stochastic matrix, converge to the steady state vector for P as k increases.

c. Let P be an $n \times n$ regular stochastic matrix, $\mathbf{q}$ the steady state vector of P, and $\mathbf{e}_j$ the j^{th} column of the $n \times n$ identity matrix. Consider the Markov chain $\{\mathbf{x}_k\}$ where $\mathbf{x}_{k+1} = P\mathbf{x}_k$ and $\mathbf{x}_0 = e_j$. By Theorem 18, $\mathbf{x}_k = P^k\mathbf{x}_0$ converges to $\mathbf{q}$ as $k \to \infty$. But $P^k\mathbf{x}_0 = P^k\mathbf{e}_j$, which is the j^{th} column of P^k. Thus the j^{th} column of P^k converges to $\mathbf{q}$ as $k \to \infty$; that is, $P^k \to [\mathbf{q} \quad \mathbf{q} \quad \ldots \quad \mathbf{q}]$.

32. Answers will vary.

MATLAB Student Version 4.0 code for Method (1):
```
A=randstoc(32); flops(0);
tic, x=nulbasis(A-eye(32));
q=x/sum(x); toc, flops
```

MATLAB Student Version 4.0 code for Method (2):
```
A=randstoc(32); flops(0);
tic, B=A^100; q=B(: ,1); toc, flops
```

Chapter 5 - Supplementary Exercises

1. True. If A is invertible and if $A\mathbf{x} = 1 \cdot \mathbf{x}$ for some nonzero $\mathbf{x}$, then left-multiply by A^{-1} to obtain $\mathbf{x} = A^{-1}\mathbf{x}$, which may be rewritten as $A^{-1}\mathbf{x} = 1 \cdot \mathbf{x}$. Since $\mathbf{x}$ is nonzero, this shows 1 is an eigenvalue of A^{-1}.
2. False. If A is row equivalent to the identity matrix, then A is invertible. The matrix in Example 4 of Section 5.3 shows that an invertible matrix need not be diagonalizable. Also, see Exercise 31 in Section 5.3.
3. True. If A contains a row or column of zeros, then A is not row equivalent to the identity matrix and thus is not invertible. By the Invertible Matrix Theorem (as stated in Section 5.2), 0 is an eigenvalue of A.
4. False. Consider a diagonal matrix D whose eigenvalues are 1 and 3, that is, its diagonal entries are 1 and 3. Then D^2 is a diagonal matrix whose eigenvalues (diagonal entries) are 1 and 9. In general, the eigenvalues of A^2 are the *squares* of the eigenvalues of A.
5. True. Suppose a nonzero vector $\mathbf{x}$ satisfies $A\mathbf{x} = \lambda\mathbf{x}$, then $A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda A\mathbf{x} = \lambda^2\mathbf{x}$

   This shows that $\mathbf{x}$ is also an eigenvector for A^2.
6. True. Suppose a nonzero vector $\mathbf{x}$ satisfies $A\mathbf{x} = \lambda\mathbf{x}$, then left-multiply by A^{-1} to obtain $\mathbf{x} = A^{-1}(\lambda\mathbf{x}) = \lambda A^{-1}\mathbf{x}$. Since A is invertible, the eigenvalue λ is not zero. So $\lambda^{-1}\mathbf{x} = A^{-1}\mathbf{x}$, which shows that $\mathbf{x}$ is also an eigenvector of A^{-1}.
7. False. Zero is an eigenvalue of each singular square matrix.
8. True. By definition, an eigenvector must be nonzero.
9. False. See Example 4 of Section 5.1.

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10. True. This follows from Theorem 4 in Section 5.2.

11. False. Let A be the 3×3 matrix in Example 3 of Section 5.3. Then A is similar to a diagonal matrix D. The eigenvectors of D are the columns of I_3, but the eigenvectors of A are entirely different.

12. False. Let $A=\begin{bmatrix}2 & 0\\ 0 & 3\end{bmatrix}$. Then $\mathbf{e}_1=\begin{bmatrix}1\\0\end{bmatrix}$ and $\mathbf{e}_2=\begin{bmatrix}0\\1\end{bmatrix}$ are eigenvectors of A, but $\mathbf{e}_1+\mathbf{e}_2$ is not.

(Actually, it can be shown that if two eigenvectors of A correspond to distinct eigenvalues, then their sum cannot be an eigenvector.)

13. False. *All* the diagonal entries of an upper triangular matrix are the eigenvalues of the matrix (Theorem 1 in Section 5.1). A diagonal entry may be zero.

14. True. Matrices A and A^T have the same characteristic polynomial, because $\det(A^T-\lambda I)=\det(A-\lambda I)^T=\det(A-\lambda I)$, by the determinant transpose property.

15. False. Counterexample: Let A be the 5×5 identity matrix.

16. True. For example, let A be the matrix that rotates vectors through $\pi/2$ radians about the origin. Then $A\mathbf{x}$ is not a multiple of $\mathbf{x}$ when $\mathbf{x}$ is nonzero.

17. False. If A is a diagonal matrix with 0 on the diagonal, then the columns of A are not linearly independent.

18. True. If $A\mathbf{x}=\lambda_1\mathbf{x}$ and $A\mathbf{x}=\lambda_2\mathbf{x}$, then $\lambda_1\mathbf{x}=\lambda_2\mathbf{x}$ and $(\lambda_1-\lambda_2)\mathbf{x}=\mathbf{0}$. If $\mathbf{x}\neq\mathbf{0}$, then λ_1 must equal λ_2.

19. False. Let A be a singular matrix that is diagonalizable. (For instance, let A be a diagonal matrix with 0 on the diagonal.) Then, by Theorem 8 in Section 5.4, the transformation $\mathbf{x}\mapsto A\mathbf{x}$ is represented by a diagonal matrix relative to a coordinate system determined by eigenvectors of A.

20. True. By definition of matrix multiplication,

$$A=AI=A[\mathbf{e}_1 \quad \mathbf{e}_2 \quad \cdots \quad \mathbf{e}_n]=[A\mathbf{e}_1 \quad A\mathbf{e}_2 \quad \cdots \quad A\mathbf{e}_n]$$

If $A\mathbf{e}_j=d_j\mathbf{e}_j$ for $j=1,\dots,n$, then A is a diagonal matrix with diagonal entries $d_1,\dots,d_n$.

21. True. If $B=PDP^{-1}$, where D is a diagonal matrix, and if $A=QBQ^{-1}$, then $A=Q(PDP^{-1})Q^{-1}=(QP)D(PQ)^{-1}$, which shows that A is diagonalizable.

22. True. Since B is invertible, AB is similar to $B(AB)B^{-1}$, which equals BA.

23. False. Having n linearly independent eigenvectors makes an $n\times n$ matrix diagonalizable (by the Diagonalization Theorem 5 in Section 5.3), but not necessarily invertible. One of the eigenvalues of the matrix could be zero.

24. Suppose $B\mathbf{x}\neq\mathbf{0}$ and $AB\mathbf{x}=\lambda\mathbf{x}$ for some λ. Then $A(B\mathbf{x})=\lambda\mathbf{x}$. Left-multiply each side by B and obtain $BA(B\mathbf{x})=B(\lambda\mathbf{x})=\lambda(B\mathbf{x})$. This equation says that $B\mathbf{x}$ is an eigenvector of BA, because $B\mathbf{x}\neq\mathbf{0}$.

25. **a**. Suppose $A\mathbf{x}=\lambda\mathbf{x}$, with $\mathbf{x}\neq\mathbf{0}$. Then $(5I-A)\mathbf{x}=5\mathbf{x}-A\mathbf{x}=5\mathbf{x}-\lambda\mathbf{x}=(5-\lambda)\mathbf{x}$. The eigenvalue is $5-\lambda$.

b. $(5I-3A+A^2)\mathbf{x}=5\mathbf{x}-3A\mathbf{x}+A(A\mathbf{x})=5\mathbf{x}-3(\lambda\mathbf{x})+\lambda^2\mathbf{x}=(5-3\lambda+\lambda^2)\mathbf{x}$. The eigenvalue is $5-3\lambda+\lambda^2$.

26. Assume that $A\mathbf{x}=\lambda\mathbf{x}$ for some nonzero vector $\mathbf{x}$. The desired statement is true for $m=1$, by the assumption about λ. Suppose that for some $k\geq 1$, the statement holds when $m=k$. That is, suppose

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that $A^k\mathbf{x} = \lambda^k\mathbf{x}$. Then $A^{k+1}\mathbf{x} = A(A^k\mathbf{x}) = A(\lambda^k\mathbf{x})$ by the induction hypothesis. Continuing, $A^{k+1}\mathbf{x} = \lambda^k A\mathbf{x} = \lambda^{k+1}\mathbf{x}$, because **x** is an eigenvector of *A* corresponding to *A*. Since **x** is nonzero, this equation shows that λ^{k+1} is an eigenvalue of A^{k+1}, with corresponding eigenvector **x**. Thus the desired statement is true when $m = k+1$. By the principle of induction, the statement is true for each positive integer *m*.

27. Suppose $A\mathbf{x} = \lambda\mathbf{x}$, with $\mathbf{x} \neq \mathbf{0}$. Then

$$\begin{aligned} p(A)\mathbf{x} &= (c_0 I + c_1 A + c_2 A^2 + \ldots + c_n A^n)\mathbf{x} \\ &= c_0\mathbf{x} + c_1 A\mathbf{x} + c_2 A^2\mathbf{x} + \ldots + c_n A^n\mathbf{x} \\ &= c_0\mathbf{x} + c_1\lambda\mathbf{x} + c_2\lambda^2\mathbf{x} + \ldots + c_n\lambda^n\mathbf{x} = p(\lambda)\mathbf{x} \end{aligned}$$

So $p(\lambda)$ is an eigenvalue of $p(A)$.

28. **a**. If $A = PDP^{-1}$, then $A^k = PD^kP^{-1}$, and

$$\begin{aligned} B &= 5I - 3A + A^2 = 5PIP^{-1} - 3PDP^{-1} + PD^2P^{-1} \\ &= P(5I - 3D + D^2)P^{-1} \end{aligned}$$

Since *D* is diagonal, so is $5I - 3D + D^2$. Thus *B* is similar to a diagonal matrix.

b.
$$\begin{aligned} p(A) &= c_0 I + c_1 PDP^{-1} + c_2 PD^2P^{-1} + \cdots + c_n PD^nP^{-1} \\ &= P(c_0 I + c_1 D + c_2 D^2 + \cdots + c_n D^n)P^{-1} \\ &= Pp(D)P^{-1} \end{aligned}$$

This shows that $p(A)$ is diagonalizable, because $p(D)$ is a linear combination of diagonal matrices and hence is diagonal. In fact, because *D* is diagonal, it is easy to see that

$$p(D) = \begin{bmatrix} p(2) & 0 \\ 0 & p(7) \end{bmatrix}.$$

29. If $A = PDP^{-1}$, then $p(A) = Pp(D)P^{-1}$, as shown in Exercise 28. If the (j, j) entry in *D* is λ, then the (j, j) entry in D^k is λ^k, and so the (j, j) entry in $p(D)$ is $p(\lambda)$. If *p* is the characteristic polynomial of *A*, then $p(\lambda) = 0$ for each diagonal entry of *D*, because these entries in *D* are the eigenvalues of *A*. Thus $p(D)$ is the zero matrix. Thus $p(A) = P \cdot 0 \cdot P^{-1} = 0$.

30. **a**. If λ is an eigenvalue of an $n \times n$ diagonalizable matrix *A*, then $A = PDP^{-1}$ for an invertible matrix *P* and an $n \times n$ diagonal matrix *D* whose diagonal entries are the eigenvalues of *A*. If the multiplicity of λ is *n*, then λ must appear in every diagonal entry of *D*. That is, $D = \lambda I$. In this case, $A = P(\lambda I)P^{-1} = \lambda PIP^{-1} = \lambda PP^{-1} = \lambda I$.

b. Since the matrix $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ is triangular, its eigenvalues are on the diagonal. Thus 3 is an eigenvalue with multiplicity 2. If the 2×2 matrix *A* were diagonalizable, then *A* would be 3*I*, by part (a). This is not the case, so *A* is not diagonalizable.

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31. If $I - A$ were not invertible, then the equation $(I - A)\mathbf{x} = \mathbf{0}$. would have a nontrivial solution $\mathbf{x}$. Then $\mathbf{x} - A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = 1 \cdot \mathbf{x}$, which shows that A would have 1 as an eigenvalue. This cannot happen if all the eigenvalues are less than 1 in magnitude. So $I - A$ must be invertible.

32. To show that A^k tends to the zero matrix, it suffices to show that each column of A^k can be made as close to the zero vector as desired by taking k sufficiently large. The jth column of A is $A\mathbf{e}_j$, where $\mathbf{e}_j$ is the jth column of the identity matrix. Since A is diagonalizable, there is a basis for $\mathbb{R}^n$ consisting of eigenvectors $\mathbf{v}_1,\ldots,\mathbf{v}_n$, corresponding to eigenvalues $\lambda_1,\ldots,\lambda_n$. So there exist scalars $c_1,\ldots,c_n$, such that $\mathbf{e}_j = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$ (an eigenvector decomposition of $\mathbf{e}_j$). Then, for $k = 1, 2, \ldots,$ the vector $A^k\mathbf{e}_j = c_1(\lambda_1)^k\mathbf{v}_1 + \cdots + c_n(\lambda_n)^k\mathbf{v}_n$. If the eigenvalues are all less than 1 in absolute value, then their kth powers all tend to zero. So the equation shows that $A^k\mathbf{e}_j$ tends to the zero vector, as desired.

33. **a**. Take $\mathbf{x}$ in H. Then $\mathbf{x} = c\mathbf{u}$ for some scalar c. So $A\mathbf{x} = A(c\mathbf{u}) = c(A\mathbf{u}) = c(\lambda\mathbf{u}) = (c\lambda)\mathbf{u}$, which shows that $A\mathbf{x}$ is in H.

b. Let $\mathbf{x}$ be a nonzero vector in K. Since K is one-dimensional, K must be the set of all scalar multiples of $\mathbf{x}$. If K is invariant under A, then $A\mathbf{x}$ is in K and hence $A\mathbf{x}$ is a multiple of $\mathbf{x}$. Thus $\mathbf{x}$ is an eigenvector of A.

34. Let U and V be echelon forms of A and B, obtained with r and s row interchanges, respectively, and no scaling. Then $\det A = (-1)^r \det U$ and $\det B = (-1)^s \det V$

Using first the row operations that reduce A to U, we can reduce G to a matrix of the form $G' = \begin{bmatrix} U & Y \\ 0 & B \end{bmatrix}$. Then, using the row operations that reduce B to V, we can further reduce G' to $G'' = \begin{bmatrix} U & Y \\ 0 & V \end{bmatrix}$. There will be $r + s$ row interchanges, and so

$\det G = \det \begin{bmatrix} A & X \\ 0 & B \end{bmatrix} = (-1)^{r+s} \det \begin{bmatrix} U & Y \\ 0 & V \end{bmatrix}$ Since $\begin{bmatrix} U & Y \\ 0 & V \end{bmatrix}$ is upper triangular, its determinant equals the product of the diagonal entries, and since U and V are upper triangular, this product also equals (det U) (det V). Thus $\det G = (-1)^{r+s}(\det U)(\det V) = (\det A)(\det B)$.

For any scalar λ, the matrix $G - \lambda I$ has the same partitioned form as G, with $A - \lambda I$ and $B - \lambda I$ as its diagonal blocks. (Here I represents various identity matrices of appropriate sizes.) Hence the result about det G shows that $\det(G - \lambda I) = \det(A - \lambda I) \cdot \det(B - \lambda I)$

35. By Exercise 34, the eigenvalues of A are the eigenvalues of the matrix $[3]$ together with the eigenvalues of $\begin{bmatrix} 5 & -2 \\ -4 & 3 \end{bmatrix}$. The only eigenvalue of $[3]$ is 3, while the eigenvalues of $\begin{bmatrix} 5 & -2 \\ -4 & 3 \end{bmatrix}$ are 1 and 7. Thus the eigenvalues of A are 1, 3, and 7.

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36. By Exercise 34, the eigenvalues of A are the eigenvalues of the matrix $\begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}$ together with the eigenvalues of $\begin{bmatrix} -7 & -4 \\ 3 & 1 \end{bmatrix}$. The eigenvalues of $\begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}$ are -1 and 6, while the eigenvalues of $\begin{bmatrix} -7 & -4 \\ 3 & 1 \end{bmatrix}$ are -5 and -1. Thus the eigenvalues of A are $-1, -5,$ and 6, and the eigenvalue -1 has multiplicity 2.

37. Replace A by $A-\lambda$ in the determinant formula from Exercise 30 in Chapter 3 Supplementary Exercises.

$\det(A-\lambda I) = (a-b-\lambda)^{n-1}[a-\lambda+(n-1)b]$

This determinant is zero only if $a-b-\lambda = 0$ or $a-\lambda+(n-1)b = 0$. Thus λ is an eigenvalue of A if and only if $\lambda = a-b$ or $\lambda = a+(n-1)b$. From the formula for $\det(A-\lambda I)$ above, the algebraic multiplicity is $n-1$ for $a-b$ and 1 for $a+(n-1)b$.

38. The 3×3 matrix has eigenvalues $1-2$ and $1+(2)(2)$, that is, -1 and 5. The eigenvalues of the 5×5 matrix are $7-3$ and $7+(4)(3)$, that is 4 and 19.

39. Note that $\det(A-\lambda I) = (a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21} = \lambda^2 - (a_{11}+a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21})$
$= \lambda^2 - (\text{tr } A)\lambda + \det A,$ and use the quadratic formula to solve the characteristic equation:

$$\lambda = \frac{\text{tr } A \pm \sqrt{(\text{tr } A)^2 - 4\det A}}{2}$$

The eigenvalues are both real if and only if the discriminant is nonnegative, that is, $(\text{tr } A)^2 - 4\det A \geq 0.$ This inequality simplifies to $(\text{tr } A)^2 \geq 4\det A$ and $\left(\frac{\text{tr}A}{2}\right)^2 \geq \det A.$

40. The eigenvalues of A are 1 and .6. Use this to factor A and A^k.

$$A = \begin{bmatrix} -1 & -3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & .6 \end{bmatrix} \cdot \frac{1}{4}\begin{bmatrix} 2 & 3 \\ -2 & -1 \end{bmatrix}$$

$$A^k = \begin{bmatrix} -1 & -3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1^k & 0 \\ 0 & .6^k \end{bmatrix} \cdot \frac{1}{4}\begin{bmatrix} 2 & 3 \\ -2 & -1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -1 & -3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -2\cdot(.6)^k & -(.6)^k \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} -2+6(.6)^k & -3+3(.6)^k \\ 4-4(.6)^k & 6-2(.6)^k \end{bmatrix}$$

$$\to \frac{1}{4}\begin{bmatrix} -2 & -3 \\ 4 & 6 \end{bmatrix} \text{ as } k \to \infty$$

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41. $C_p = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}$; $\det(C_p - \lambda I) = 6 - 5\lambda + \lambda^2 = p(\lambda)$

42. $C_p = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 24 & -26 & 9 \end{bmatrix}$; $\det(C_p - \lambda I) = 24 - 26\lambda + 9\lambda^2 - \lambda^3 = p(\lambda)$.

43. If p is a polynomial of order 2, then a calculation such as in Exercise 41 shows that the characteristic polynomial of C_p is $p(\lambda) = (-1)^2 p(\lambda)$, so the result is true for $n = 2$. Suppose the result is true for $n = k$ for some $k \geq 2$, and consider a polynomial p of degree $k+1$. Then expanding $\det(C_p - \lambda I)$ by cofactors down the first column, the determinant of $C_p - \lambda I$ equals

$$(-\lambda)\det\begin{bmatrix} -\lambda & 1 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & & & 1 \\ -a_1 & -a_2 & \cdots & -a_k - \lambda \end{bmatrix} + (-1)^{k+1} a_0$$

The $k \times k$ matrix shown is $C_q - \lambda I$, where $q(t) = a_1 + a_2 t + \cdots + a_k t^{k-1} + t^k$. By the induction assumption, the determinant of $C_q - \lambda I$ is $(-1)^k q(\lambda)$. Thus

$$\begin{aligned} \det(C_p - \lambda I) &= (-1)^{k+1} a_0 + (-\lambda)(-1)^k q(\lambda) \\ &= (-1)^{k+1}[a_0 + \lambda(a_1 + \cdots + a_k \lambda^{k-1} + \lambda^k)] \\ &= (-1)^{k+1} p(\lambda) \end{aligned}$$

So the formula holds for $n = k+1$ when it holds for $n = k$. By the principle of induction, the formula for $\det(C_p - \lambda I)$ is true for all $n \geq 2$.

44. **a**. $C_p = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix}$

b. Since λ is a zero of p, $a_0 + a_1\lambda + a_2\lambda^2 + \lambda^3 = 0$ and $-a_0 - a_1\lambda - a_2\lambda^2 = \lambda^3$. Thus $C_p \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda^2 \\ -a_0 - a_1\lambda - a_2\lambda^2 \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda^2 \\ \lambda^3 \end{bmatrix}$. That is, $C_p(1, \lambda, \lambda^2) = \lambda(1, \lambda, \lambda^2)$, which shows that $(1, \lambda, \lambda^2)$ is an eigenvector of C_p corresponding to the eigenvalue λ.

45. From Exercise 44, the columns of the Vandermonde matrix V are eigenvectors of C_p, corresponding to the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ (the roots of the polynomial p). Since these eigenvalues are distinct, the eigenvectors from a linearly independent set, by Theorem 2 in Section 5.1. Thus V has linearly independent columns and hence is invertible, by the Invertible Matrix Theorem. Finally, since the columns of V are eigenvectors of C_p, the Diagonalization Theorem (Theorem 5 in Section 5.3) shows that $V^{-1}C_pV$ is diagonal.

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46. The MATLAB command `roots (p)` requires as input a row vector p whose entries are the coefficients of a polynomial, with the highest order coefficient listed first. MATLAB constructs a companion matrix C_p whose characteristic polynomial is p, so the roots of p are the eigenvalues of C_p. The numerical values of the eigenvalues (roots) are found by the same QR algorithm used by the command `eig(A)`.

47. The MATLAB command `[P D] = eig(A)` produces a matrix P, whose condition number is 1.6×10^8, and a diagonal matrix D, whose entries are *almost* 2, 2, 1. However, the exact eigenvalues of A are 2, 2, 1, and A is not diagonalizable.

48. This matrix may cause the same sort of trouble as the matrix in Exercise 47. A matrix program that computes eigenvalues by an interative process may indicate that A has four distinct eigenvalues, all close to zero. However, the only eigenvalue is 0, with multiplicity 4, because $A^4 = 0$.

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