7 Symmetric Matrices and Quadratic Forms

7.1 - Diagonalization of Symmetric Matrices

Notes: Students can profit by reviewing Section 5.3 (focusing on the Diagonalization Theorem) before working on this section. Theorems 1 and 2 and the calculations in Examples 2 and 3 are important for the sections that follow. Note that *symmetric matrix* means *real symmetric matrix*, because all matrices in the text have real entries, as mentioned at the beginning of this chapter. The exercises in this section have been constructed so that mastery of the Gram-Schmidt process is not needed.

Theorem 2 is easily proved for the 2×2 case:

$$\text{If } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ then } \lambda = \frac{1}{2}\left(a + d \pm \sqrt{(a-d)^2 + 4b^2}\right).$$

If $b = 0$ there is nothing to prove. Otherwise, there are two distinct eigenvalues, so A must be diagonalizable. In each case, an eigenvector for λ is $\begin{bmatrix} d-\lambda \\ -b \end{bmatrix}$.

1. Since $A = \begin{bmatrix} 3 & 5 \\ 5 & -7 \end{bmatrix} = A^T$, the matrix is symmetric.

2. Since $A = \begin{bmatrix} 3 & -5 \\ -5 & -3 \end{bmatrix} = A^T$, the matrix is symmetric.

3. Since $A = \begin{bmatrix} 2 & 3 \\ 4 & 4 \end{bmatrix} \neq A^T$, the matrix is not symmetric.

4. Since $A = \begin{bmatrix} 0 & 8 & 3 \\ 8 & 0 & -4 \\ 3 & 2 & 0 \end{bmatrix} \neq A^T$, the matrix is not symmetric.

5. Since $A = \begin{bmatrix} -6 & 2 & 0 \\ 2 & -6 & 2 \\ 0 & 2 & -6 \end{bmatrix} = A^T$, the matrix is symmetric.

6. Since A is not a square matrix $A \neq A^T$ and the matrix is not symmetric.

7. Let $P = \begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}$, and compute that $P^T P = \begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}\begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$. Since P is a square matrix, P is orthogonal and $P^{-1} = P^T = \begin{bmatrix} .6 & .8 \\ .8 & -.6 \end{bmatrix}$.

8. Let $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, and compute that $P^T P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2I_2 \neq I_2$. Thus P is not orthogonal.

9. Let $P = \begin{bmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{bmatrix}$, and compute that $P^T P = \begin{bmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{bmatrix}\begin{bmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$.
Since P is a square matrix, P is orthogonal and $P^{-1} = P^T = \begin{bmatrix} -4/5 & 3/5 \\ 3/5 & 4/5 \end{bmatrix}$.

10. Let $P = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$, and compute that

$$P^T P = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3.$$

Since P is a square matrix, P is orthogonal and $P^{-1} = P^T = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$.

11. Let $P = \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ 0 & 1/3 & -2/3 \\ 5/3 & -4/3 & -2/3 \end{bmatrix}$, and compute that

$$P^T P = \begin{bmatrix} 2/3 & 0 & 5/3 \\ 2/3 & 1/3 & -4/3 \\ 1/3 & -2/3 & -2/3 \end{bmatrix}\begin{bmatrix} 2/3 & 2/3 & 1/3 \\ 0 & 1/3 & -2/3 \\ 5/3 & -4/3 & -2/3 \end{bmatrix} = \begin{bmatrix} 29/9 & -16/9 & -8/9 \\ -16/9 & 21/9 & 8/9 \\ -8/9 & 8/9 & 1 \end{bmatrix} \neq I_3.$$

Thus P is not orthogonal.

12. Let $P = \begin{bmatrix} .5 & .5 & -.5 & -.5 \\ .5 & .5 & .5 & .5 \\ .5 & -.5 & -.5 & .5 \\ .5 & -.5 & .5 & -.5 \end{bmatrix}$, and compute that

$P^T P = \begin{bmatrix} .5 & .5 & .5 & .5 \\ .5 & .5 & -.5 & -.5 \\ -.5 & .5 & -.5 & .5 \\ -.5 & .5 & .5 & -.5 \end{bmatrix} \begin{bmatrix} .5 & .5 & -.5 & -.5 \\ .5 & .5 & .5 & .5 \\ .5 & -.5 & -.5 & .5 \\ .5 & -.5 & .5 & -.5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I_4$. Since P is a square matrix, P is orthogonal and $P^{-1} = P^T = \begin{bmatrix} .5 & .5 & .5 & .5 \\ .5 & .5 & -.5 & -.5 \\ -.5 & .5 & -.5 & .5 \\ -.5 & .5 & .5 & -.5 \end{bmatrix}$.

13. Let $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$. Then the characteristic polynomial of A is $(3-\lambda)^2 - 1 = \lambda^2 - 6\lambda + 8 = (\lambda - 4)(\lambda - 2)$, so the eigenvalues of A are 4 and 2. For $\lambda = 4$, one computes that a basis for the eigenspace is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, which can be normalized to get $\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$. For $\lambda = 2$ one computes that a basis for the eigenspace is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, which can be normalized to get $\mathbf{u}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$. Let $P = [\mathbf{u}_1 \quad \mathbf{u}_2] = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$. Then P orthogonally diagonalizes A, and $A = PDP^{-1}$.

14. Let $A = \begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix}$. Then the characteristic polynomial of A is $(1-\lambda)^2 - 25 = \lambda^2 - 2\lambda - 24$ $= (\lambda - 6)(\lambda + 4)$, so the eigenvalues of A are 6 and –4. For $\lambda = 6$, one computes that a basis for the eigenspace is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, which can be normalized to get $\mathbf{u}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$. For $\lambda = -4$, one computes that a basis for the eigenspace is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, which can be normalized to get $\mathbf{u}_2 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$. Let $P = [\mathbf{u}_1 \quad \mathbf{u}_2] = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ and $D = \begin{bmatrix} 6 & 0 \\ 0 & -4 \end{bmatrix}$. Then P orthogonally diagonalizes A, and $A = PDP^{-1}$.

15. Let $A = \begin{bmatrix} 3 & 4 \\ 4 & 9 \end{bmatrix}$. Then the characteristic polynomial of A is $(3-\lambda)(9-\lambda) - 16 = \lambda^2 - 12\lambda + 11$ $= (\lambda - 11)(\lambda - 1)$, so the eigenvalues of A are 11 and 1. For $\lambda = 11$, one computes that a basis for the eigenspace is $\begin{bmatrix}1\\2\end{bmatrix}$, which can be normalized to get $\mathbf{u}_1 = \begin{bmatrix}1/\sqrt{5}\\2/\sqrt{5}\end{bmatrix}$. For $\lambda = 1$, one computes that a basis for the eigenspace is $\begin{bmatrix}-2\\1\end{bmatrix}$, which can be normalized to get $\mathbf{u}_2 = \begin{bmatrix}-2/\sqrt{5}\\1/\sqrt{5}\end{bmatrix}$. Let $P = [\mathbf{u}_1 \quad \mathbf{u}_2] = \begin{bmatrix}1/\sqrt{5} & -2/\sqrt{5}\\2/\sqrt{5} & 1/\sqrt{5}\end{bmatrix}$ and $D = \begin{bmatrix}11 & 0\\0 & 1\end{bmatrix}$. Then P orthogonally diagonalizes A, and $A = PDP^{-1}$.

16. Let $A = \begin{bmatrix}6 & -2\\-2 & 9\end{bmatrix}$. Then the characteristic polynomial of A is $(6-\lambda)(9-\lambda)-4 = \lambda^2 - 15\lambda + 50$ $= (\lambda-5)(\lambda-10)$, so the eigenvalues of A are 5 and 10. For $\lambda = 5$, one computes that a basis for the eigenspace is $\begin{bmatrix}2\\1\end{bmatrix}$, which can be normalized to get $\mathbf{u}_1 = \begin{bmatrix}2/\sqrt{5}\\1/\sqrt{5}\end{bmatrix}$. For $\lambda = 10$, one computes that a basis for the eigenspace is $\begin{bmatrix}-1\\2\end{bmatrix}$, which can be normalized to get $\mathbf{u}_2 = \begin{bmatrix}-1/\sqrt{5}\\2/\sqrt{5}\end{bmatrix}$. Let $P = [\mathbf{u}_1 \quad \mathbf{u}_2] = \begin{bmatrix}2/\sqrt{5} & -1/\sqrt{5}\\1/\sqrt{5} & 2/\sqrt{5}\end{bmatrix}$ and $D = \begin{bmatrix}5 & 0\\0 & 10\end{bmatrix}$. Then P orthogonally diagonalizes A, and $A = PDP^{-1}$.

17. Let $A = \begin{bmatrix}1 & 1 & 5\\1 & 5 & 1\\5 & 1 & 1\end{bmatrix}$. The eigenvalues of A are -4, 4, and 7. For $\lambda = -4$, one computes that a basis for the eigenspace is $\begin{bmatrix}-1\\0\\1\end{bmatrix}$, which can be normalized to get $\mathbf{u}_1 = \begin{bmatrix}-1/\sqrt{2}\\0\\1/\sqrt{2}\end{bmatrix}$. For $\lambda = 4$, one computes that a basis for the eigenspace is $\begin{bmatrix}1\\-2\\1\end{bmatrix}$, which can be normalized to get $\mathbf{u}_2 = \begin{bmatrix}1/\sqrt{6}\\-2/\sqrt{6}\\1/\sqrt{6}\end{bmatrix}$. For $\lambda = 7$, one computes that a basis for the eigenspace is $\begin{bmatrix}1\\1\\1\end{bmatrix}$, which can be normalized to get $\mathbf{u}_3 = \begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}$. Let $P = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3] = \begin{bmatrix}-1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3}\\0 & -2/\sqrt{6} & 1/\sqrt{3}\\1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3}\end{bmatrix}$ and $D = \begin{bmatrix}-4 & 0 & 0\\0 & 4 & 0\\0 & 0 & 7\end{bmatrix}$. Then P orthogonally diagonalizes A, and $A = PDP^{-1}$.

18. Let $A=\begin{bmatrix}1 & -6 & 4\\ -6 & 2 & -2\\ 4 & -2 & -3\end{bmatrix}$. The eigenvalues of A are -3, -6 and 9. For $\lambda=-3$, one computes that a basis for the eigenspace is $\begin{bmatrix}1\\2\\2\end{bmatrix}$, which can be normalized to get $\mathbf{u}_1=\begin{bmatrix}1/3\\2/3\\2/3\end{bmatrix}$. For $\lambda=-6$, one computes that a basis for the eigenspace is $\begin{bmatrix}-2\\-1\\2\end{bmatrix}$, which can be normalized to get $\mathbf{u}_2=\begin{bmatrix}-2/3\\-1/3\\2/3\end{bmatrix}$. For $\lambda=9$, one computes that a basis for the eigenspace is $\begin{bmatrix}2\\-2\\1\end{bmatrix}$, which can be normalized to get $\mathbf{u}_3=\begin{bmatrix}2/3\\-2/3\\1/3\end{bmatrix}$. Let $P=[\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3]=\begin{bmatrix}1/3 & -2/3 & 2/3\\ 2/3 & -1/3 & -2/3\\ 2/3 & 2/3 & 1/3\end{bmatrix}$ and $D=\begin{bmatrix}-3 & 0 & 0\\ 0 & -6 & 0\\ 0 & 0 & 9\end{bmatrix}$. Then P orthogonally diagonalizes A, and $A=PDP^{-1}$.

19. Let $A=\begin{bmatrix}3 & -2 & 4\\ -2 & 6 & 2\\ 4 & 2 & 3\end{bmatrix}$. The eigenvalues of A are 7 and –2. For $\lambda=7$, one computes that a basis for the eigenspace is $\left\{\begin{bmatrix}-1\\2\\0\end{bmatrix},\begin{bmatrix}1\\0\\1\end{bmatrix}\right\}$. This basis may be converted via orthogonal projection to an orthogonal basis for the eigenspace: $\left\{\begin{bmatrix}-1\\2\\0\end{bmatrix},\begin{bmatrix}4\\2\\5\end{bmatrix}\right\}$. These vectors can be normalized to get $\mathbf{u}_1=\begin{bmatrix}-1/\sqrt{5}\\2/\sqrt{5}\\0\end{bmatrix}$, $\mathbf{u}_2=\begin{bmatrix}4/\sqrt{45}\\2/\sqrt{45}\\5/\sqrt{45}\end{bmatrix}$. For $\lambda=-2$, one computes that a basis for the eigenspace is $\begin{bmatrix}-2\\-1\\2\end{bmatrix}$, which can be normalized to get $\mathbf{u}_3=\begin{bmatrix}-2/3\\-1/3\\2/3\end{bmatrix}$. Let $P=[\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3]=\begin{bmatrix}-1/\sqrt{5} & 4/\sqrt{45} & -2/3\\ 2/\sqrt{5} & 2/\sqrt{45} & -1/3\\ 0 & 5/\sqrt{45} & 2/3\end{bmatrix}$ and $D=\begin{bmatrix}7 & 0 & 0\\ 0 & 7 & 0\\ 0 & 0 & -2\end{bmatrix}$. Then P orthogonally diagonalizes A, and $A=PDP^{-1}$.

20. Let $A = \begin{bmatrix} 5 & 8 & -4 \\ 8 & 5 & -4 \\ -4 & -4 & -1 \end{bmatrix}$. The eigenvalues of A are -3 and 15. For $\lambda = -3$, one computes that a basis for the eigenspace is $\left\{ \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \right\}$ which is orthogonal and can be normalized to get $\{\mathbf{u}_1, \mathbf{u}_2\} = \left\{ \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \end{bmatrix} \right\}$. For $\lambda = 15$, one computes that a basis for the eigenspace is $\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$, which can be normalized to get $\mathbf{u}_3 = \begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix}$. Let $P = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3] = \begin{bmatrix} 2/3 & -1/3 & 2/3 \\ -1/3 & 2/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix}$ and $D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$. Then P orthogonally diagonalizes A, and $A = PDP^{-1}$.

21. Let $A = \begin{bmatrix} 4 & 3 & 1 & 1 \\ 3 & 4 & 1 & 1 \\ 1 & 1 & 4 & 3 \\ 1 & 1 & 3 & 4 \end{bmatrix}$. The eigenvalues of A are 1, 5, and 9. For $\lambda = 1$, one computes that a basis for the eigenspace is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\}$ which is an orthogonal set and can be normalized to get $\{\mathbf{u}_1, \mathbf{u}_2\} = \left\{ \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}$. For $\lambda = 5$, one computes that a basis for the eigenspace is $\begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$, which can be normalized to get $\mathbf{u}_3 = \begin{bmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$. For $\lambda = 9$, one computes that a basis for the eigenspace is $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$. This vector can be normalized to get $\mathbf{u}_4 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$. Let

$P = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3 \quad \mathbf{u}_4] = \begin{bmatrix} -1/\sqrt{2} & 0 & -1/2 & 1/2 \\ 1/\sqrt{2} & 0 & -1/2 & 1/2 \\ 0 & -1/\sqrt{2} & 1/2 & 1/2 \\ 0 & 1/\sqrt{2} & 1/2 & 1/2 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$. Then P orthogonally diagonalizes A, and $A = PDP^{-1}$.

22. Let $A = \begin{bmatrix} 4 & 0 & 1 & 0 \\ 0 & 4 & 0 & 1 \\ 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 4 \end{bmatrix}$. The eigenvalues of A are 3 and 5. For $\lambda = 3$, one computes that a basis for the eigenspace is $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$. This basis is an orthogonal basis for the eigenspace, and these vectors can be normalized to get $\{\mathbf{u}_1, \mathbf{u}_2\} = \left\{ \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \right\}$. For $\lambda = 5$, one computes that a basis for the eigenspace is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$, which is orthogonal and can be normalized to get $\{\mathbf{u}_3, \mathbf{u}_4\} = \left\{ \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \right\}$. Let $P = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3 \quad \mathbf{u}_4] = \begin{bmatrix} -1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$. Then P orthogonally diagonalizes A, and $A = PDP^{-1}$.

23. Let $A = \begin{bmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix}$. Since each row of A sums to 2, $A\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 & -1 & -1 \\ -1 & 4 & -1 \\ -1 & -1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector of A with corresponding eigenvalue $\lambda = 2$. The eigenvector may be normalized to get $\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$. For $\lambda = 5$, one computes that a basis for the eigenspace is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$, so $\lambda = 5$ is an eigenvalue of A. This basis may be converted via orthogonal projection to an orthogonal basis $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\}$ for the eigenspace, and these vectors can be normalized to get $\mathbf{u}_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$ and $\mathbf{u}_3 = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$. Let

$P = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3] = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. Then P orthogonally diagonalizes A, and $A = PDP^{-1}$.

24. Let $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$. One may compute that $A\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, so $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector of A with associated eigenvalue $\lambda_1 = 1$. For $\lambda_1 = 1$, one computes that a basis for the eigenspace is $\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$. This basis may be converted via orthogonal projection to an orthogonal basis for the eigenspace: $\{\mathbf{v}_1, \mathbf{v}_3\} = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}$. Likewise one may compute that $A\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, so $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is an eigenvector of A with associated eigenvalue 4. The eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ may be normalized to get the vectors $\mathbf{u}_1 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$, and $\mathbf{u}_3 = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$. Let

$P=\begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3\end{bmatrix}=\begin{bmatrix}-1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6}\\ 0 & -1/\sqrt{3} & 2/\sqrt{6}\\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6}\end{bmatrix}$ and $D=\begin{bmatrix}1 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 1\end{bmatrix}$. Then P orthogonally diagonalizes A, and $A=PDP^{-1}$.

25. True. See Theorem 2 and the paragraph preceding the theorem.

26. False. See Theorem 2.

27. False. An orthogonal matrix can be symmetric (and hence orthogonally diagonalizable), but not every orthogonal matrix is symmetric. See the matrix P in Example 2.

28. True. See the displayed equation in the paragraph before Theorem 2.

29. False. See the paragraph following formula (2), in which each **u** is a unit vector.

30. True. This is a particular case of the statement in Theorem 1, where **u** and **v** are nonzero.

31. False. There are n real eigenvalues (Theorem 3), but they need not be distinct (Example 3).

32. False. See Theorem 3(b).

33. Let A be an $n\times n$ symmetric matrix. Then $(A\mathbf{x})\cdot\mathbf{y}=(A\mathbf{x})^T\mathbf{y}=\mathbf{x}^TA^T\mathbf{y}=\mathbf{x}^TA\mathbf{y}=\mathbf{x}\cdot(A\mathbf{y})$, since $A^T=A$.

34. Since A is symmetric, $(B^TAB)^T=B^TA^TB^{TT}=B^TAB$, and B^TAB is symmetric. Applying this result with $A=I$ gives B^TB is symmetric. Finally, $(BB^T)^T=B^{TT}B^T=BB^T$, so BB^T is symmetric.

35. Since A is orthogonally diagonalizable, $A=PDP^{-1}$, where P is orthogonal and D is diagonal. Since A is invertible, $A^{-1}=(PDP^{-1})^{-1}=PD^{-1}P^{-1}$. Notice that D^{-1} is a diagonal matrix, so A^{-1} is orthogonally diagonalizable.

36. If A and B are orthogonally diagonalizable, then A and B are symmetric by Theorem 2. If $AB=BA$, then $(AB)^T=(BA)^T=A^TB^T=AB$. So AB is symmetric and hence is orthogonally diagonalizable by Theorem 2.

37. The Diagonalization Theorem of Section 5.3 says that the columns of P are linearly independent eigenvectors corresponding to the eigenvalues of A listed on the diagonal of D. So P has exactly k columns of eigenvectors corresponding to λ. These k columns form a basis for the eigenspace.

38. If $A=PRP^{-1}$, then $P^{-1}AP=R$. Since P is orthogonal, $R=P^TAP$. Hence $R^T=(P^TAP)^T=P^TA^TP^{TT}=P^TAP=R$, which shows that R is symmetric. Since R is also upper triangular, its entries above the diagonal must be zeros to match the zeros below the diagonal. Thus R is a diagonal matrix.

39. It has previously been found that A is orthogonally diagonalized by P, where

$$P=[\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3]=\begin{bmatrix} -1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \text{ and } D=\begin{bmatrix} 8 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$ Thus the spectral decomposition of A is

$$A=\lambda_1\mathbf{u}_1\mathbf{u}_1^T+\lambda_2\mathbf{u}_2\mathbf{u}_2^T+\lambda_3\mathbf{u}_3\mathbf{u}_3^T=8\mathbf{u}_1\mathbf{u}_1^T+6\mathbf{u}_2\mathbf{u}_2^T+3\mathbf{u}_3\mathbf{u}_3^T$$

$$=8\begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}+6\begin{bmatrix} 1/6 & 1/6 & -2/6 \\ 1/6 & 1/6 & -2/6 \\ -2/6 & -2/6 & 4/6 \end{bmatrix}+3\begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}.$$

40. It has previously been found that A is orthogonally diagonalized by P, where

$$P=[\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3]=\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{18} & -2/3 \\ 0 & 4/\sqrt{18} & -1/3 \\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3 \end{bmatrix} \text{ and } D=\begin{bmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Thus the spectral decomposition of A is

$$A=\lambda_1\mathbf{u}_1\mathbf{u}_1^T+\lambda_2\mathbf{u}_2\mathbf{u}_2^T+\lambda_3\mathbf{u}_3\mathbf{u}_3^T=7\mathbf{u}_1\mathbf{u}_1^T+7\mathbf{u}_2\mathbf{u}_2^T-2\mathbf{u}_3\mathbf{u}_3^T$$

$$=7\begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 0 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}+7\begin{bmatrix} 1/18 & -4/18 & -1/18 \\ -4/18 & 16/18 & 4/18 \\ -1/18 & 4/18 & 1/18 \end{bmatrix}-2\begin{bmatrix} 4/9 & 2/9 & -4/9 \\ 2/9 & 1/9 & -2/9 \\ -4/9 & -2/9 & 4/9 \end{bmatrix}$$

41. **a**. Given $\mathbf{x}$ in $\mathbb{R}^n$, $b\mathbf{x}=(\mathbf{u}\mathbf{u}^T)\mathbf{x}=\mathbf{u}(\mathbf{u}^T\mathbf{x})=(\mathbf{u}^T\mathbf{x})\mathbf{u},$ because $\mathbf{u}^T\mathbf{x}$ is a scalar. So $B\mathbf{x} = (\mathbf{x} \cdot \mathbf{u})\mathbf{u}$. Since $\mathbf{u}$ is a unit vector, $B\mathbf{x}$ is the orthogonal projection of $\mathbf{x}$ onto $\mathbf{u}$.

b. Since $B^T=(\mathbf{u}\mathbf{u}^T)^T=\mathbf{u}^{TT}\mathbf{u}^T=\mathbf{u}\mathbf{u}^T=B,$ B is a symmetric matrix. Also, $B^2=(\mathbf{u}\mathbf{u}^T)(\mathbf{u}\mathbf{u}^T)=\mathbf{u}(\mathbf{u}^T\mathbf{u})\mathbf{u}^T=\mathbf{u}\mathbf{u}^T=B$ because $\mathbf{u}^T\mathbf{u}=1$.

c. Since $\mathbf{u}^T\mathbf{u}=1$, $B\mathbf{u}=(\mathbf{u}\mathbf{u}^T)\mathbf{u}=\mathbf{u}(\mathbf{u}^T\mathbf{u})=\mathbf{u}(1)=\mathbf{u}$, so $\mathbf{u}$ is an eigenvector of B with corresponding eigenvalue 1.

42. Given any $\mathbf{y}$ in $\mathbb{R}^n$, let $\hat{\mathbf{y}}=B\mathbf{y}$ and $\mathbf{z}=\mathbf{y}-\hat{\mathbf{y}}$. Suppose that $B^T=B$ and $B^2=B$. Then $B^TB=BB=B$.

a. Since $\mathbf{z}\cdot\hat{\mathbf{y}}=(\mathbf{y}-\hat{\mathbf{y}})\cdot(B\mathbf{y})=\mathbf{y}\cdot(B\mathbf{y})-\hat{\mathbf{y}}\cdot(B\mathbf{y})=\mathbf{y}^TB\mathbf{y}-(B\mathbf{y})^TB\mathbf{y}=\mathbf{y}^TB\mathbf{y}-\mathbf{y}^TB^TB\mathbf{y}=0$, $\mathbf{z}$ is orthogonal to $\hat{\mathbf{y}}$.

b. Any vector in $W=\text{Col}\, B$ has the form $B\mathbf{u}$ for some $\mathbf{u}$. Noting that B is symmetric, Exercise 34 gives $(\mathbf{y}-\hat{\mathbf{y}})\cdot(B\mathbf{u})=[B(\mathbf{y}-\hat{\mathbf{y}})]\cdot\mathbf{u}=[B\mathbf{y}-BB\mathbf{y}]\cdot\mathbf{u}=0$, since $B^2=B$. So $\mathbf{y}-\hat{\mathbf{y}}$ is in $W^\perp$, and the decomposition $\mathbf{y}=\hat{\mathbf{y}}+(\mathbf{y}-\hat{\mathbf{y}})$ expresses $\mathbf{y}$ as the sum of a vector in W and a vector in $W^\perp$. By the Orthogonal Decomposition Theorem in Section 6.3, this decomposition is unique, and so $\hat{\mathbf{y}}$ must be $\text{proj}_W\,\mathbf{y}$.

43. Let $A=\begin{bmatrix} 6 & 2 & 9 & -6 \\ 2 & 6 & -6 & 9 \\ 9 & -6 & 6 & 2 \\ -6 & 9 & 2 & 6 \end{bmatrix}$. The eigenvalues of A are 19, 11, 5, and –11. For $\lambda=19$, one computes that a basis for the eigenspace is $\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$, which can be normalized to get $\mathbf{u}_1=\begin{bmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$. For $\lambda=11$, one computes that a basis for the eigenspace is $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, which can be normalized to get $\mathbf{u}_2=\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$. For $\lambda=5$, one computes that a basis for the eigenspace is $\begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$, which can be normalized to get $\mathbf{u}_3=\begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}$. For $\lambda=-11$, one computes that a basis for the eigenspace is $\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$, which can be normalized to get $\mathbf{u}_4=\begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$. Let

$P=[\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3 \quad \mathbf{u}_4]=\begin{bmatrix} -1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & 1/2 \end{bmatrix}$ and $D=\begin{bmatrix} 19 & 0 & 0 & 0 \\ 0 & 11 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & -11 \end{bmatrix}$. Then P orthogonally diagonalizes A, and $A=PDP^{-1}$.

44. Let $A=\begin{bmatrix} .63 & -.18 & -.06 & -.04 \\ -.18 & .84 & -.04 & .12 \\ -.06 & -.04 & .72 & -.12 \\ -.04 & .12 & -.12 & .66 \end{bmatrix}$. The eigenvalues of A are .5, .55, .8, and 1. For $\lambda=.5$, one computes that a basis for the eigenspace is $\begin{bmatrix} 4 \\ 2 \\ 2 \\ 1 \end{bmatrix}$, which can be normalized to get $\mathbf{u}_1=\begin{bmatrix} .8 \\ .4 \\ .4 \\ .2 \end{bmatrix}$. For

$\lambda = .55$, one computes that a basis for the eigenspace is $\begin{bmatrix} -1 \\ -2 \\ 2 \\ 4 \end{bmatrix}$, which can be normalized to get $\mathbf{u}_2 = \begin{bmatrix} -.2 \\ -.4 \\ .4 \\ .8 \end{bmatrix}$. For $\lambda = .8$, one computes that a basis for the eigenspace is $\begin{bmatrix} 2 \\ -1 \\ -4 \\ 2 \end{bmatrix}$, which can be normalized to get $\mathbf{u}_3 = \begin{bmatrix} .4 \\ -.2 \\ -.8 \\ .4 \end{bmatrix}$. For $\lambda = 1$, one computes that a basis for the eigenspace is $\begin{bmatrix} -2 \\ 4 \\ -1 \\ 2 \end{bmatrix}$, which can be normalized to get $\mathbf{u}_4 = \begin{bmatrix} -.4 \\ .8 \\ -.2 \\ .4 \end{bmatrix}$. Let $P = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3 \quad \mathbf{u}_4] = \begin{bmatrix} .8 & -.2 & .4 & -.4 \\ .4 & -.4 & -.2 & .8 \\ .4 & .4 & -.8 & -.2 \\ .2 & .8 & .4 & .4 \end{bmatrix}$ and $D = \begin{bmatrix} .5 & 0 & 0 & 0 \\ 0 & .55 & 0 & 0 \\ 0 & 0 & .8 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Then P orthogonally diagonalizes A, and $A = PDP^{-1}$.

45. Let $A = \begin{bmatrix} .31 & .58 & .08 & .44 \\ .58 & -.56 & .44 & -.58 \\ .08 & .44 & .19 & -.08 \\ .44 & -.58 & -.08 & .31 \end{bmatrix}$. The eigenvalues of A are .75, 0, and –1.25. For $\lambda = .75$, one computes that a basis for the eigenspace is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 2 \\ 0 \end{bmatrix} \right\}$. This basis may be converted via orthogonal projection to the orthogonal basis $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 4 \\ -3 \end{bmatrix} \right\}$. These vectors can be normalized to get $\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 3/\sqrt{50} \\ 4/\sqrt{50} \\ 4/\sqrt{50} \\ -3/\sqrt{50} \end{bmatrix}$. For $\lambda = 0$, one computes that a basis for the eigenspace is $\begin{bmatrix} -2 \\ -1 \\ 4 \\ 2 \end{bmatrix}$, which can be normalized to get $\mathbf{u}_3 = \begin{bmatrix} -2/5 \\ -1/5 \\ 4/5 \\ 2/5 \end{bmatrix}$. For $\lambda = -1.25$, one computes that a basis for the eigenspace is $\begin{bmatrix} -2 \\ 4 \\ -1 \\ 2 \end{bmatrix}$, which can be normalized to get $\mathbf{u}_4 = \begin{bmatrix} -2/5 \\ 4/5 \\ -1/5 \\ 2/5 \end{bmatrix}$. Let

$$P = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3 \quad \mathbf{u}_4] = P = \begin{bmatrix} 1/\sqrt{2} & 3/\sqrt{50} & -2/5 & -2/5 \\ 0 & 4/\sqrt{50} & -1/5 & 4/5 \\ 0 & 4/\sqrt{50} & 4/5 & -1/5 \\ 1/\sqrt{2} & -3/\sqrt{50} & 2/5 & 2/5 \end{bmatrix} \text{ and } D = \begin{bmatrix} .75 & 0 & 0 & 0 \\ 0 & .75 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1.25 \end{bmatrix}$$

. Then P orthogonally diagonalizes A, and $A = PDP^{-1}$.

46. Let $A = \begin{bmatrix} 8 & 2 & 2 & -6 & 9 \\ 2 & 8 & 2 & -6 & 9 \\ 2 & 2 & 8 & -6 & 9 \\ -6 & -6 & -6 & 24 & 9 \\ 9 & 9 & 9 & 9 & -21 \end{bmatrix}$. The eigenvalues of A are 6, 30, –30, and 15. For $\lambda = 6$, one computes that a basis for the eigenspace is $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$. This basis may be converted via orthogonal projection to the orthogonal basis $\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \\ 0 \end{bmatrix} \right\}$. These vectors can be normalized to get

$\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \\ 0 \\ 0 \end{bmatrix}$. For $\lambda = 30$, one computes that a basis for the eigenspace is $\begin{bmatrix} 1 \\ 1 \\ 1 \\ -3 \\ 0 \end{bmatrix}$, which can be normalized to get $\mathbf{u}_3 = \begin{bmatrix} 1/\sqrt{12} \\ 1/\sqrt{12} \\ 1/\sqrt{12} \\ -3/\sqrt{12} \\ 0 \end{bmatrix}$. For $\lambda = -30$, one computes that a basis for the eigenspace is $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -4 \end{bmatrix}$, which can be normalized to get $\mathbf{u}_4 = \begin{bmatrix} 1/\sqrt{20} \\ 1/\sqrt{20} \\ 1/\sqrt{20} \\ 1/\sqrt{20} \\ -4/\sqrt{20} \end{bmatrix}$. For $\lambda = 15$, one computes that a basis for the eigenspace is $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, which can be normalized to get $\mathbf{u}_5 = \begin{bmatrix} 1/\sqrt{5} \\ 1/\sqrt{5} \\ 1/\sqrt{5} \\ 1/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$. Let

$$P = [\mathbf{u}_1 \quad \mathbf{u}_2 \quad \mathbf{u}_3 \quad \mathbf{u}_4 \quad \mathbf{u}_5] = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{12} & 1/\sqrt{20} & 1/\sqrt{5} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{12} & 1/\sqrt{20} & 1/\sqrt{5} \\ 0 & -2/\sqrt{6} & 1/\sqrt{12} & 1/\sqrt{20} & 1/\sqrt{5} \\ 0 & 0 & -3/\sqrt{12} & 1/\sqrt{20} & 1/\sqrt{5} \\ 0 & 0 & 0 & -4/\sqrt{20} & 1/\sqrt{5} \end{bmatrix}$$ and

$$D = \begin{bmatrix} 6 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 0 & 0 & 30 & 0 & 0 \\ 0 & 0 & 0 & -30 & 0 \\ 0 & 0 & 0 & 0 & 15 \end{bmatrix}$$. Then P orthogonally diagonalizes A, and $A = PDP^{-1}$.

7.2 - Quadratic Forms

Notes: This section can provide a good conclusion to the course, because the mathematics here is widely used in applications. For instance, Exercises 31 and 32 can be used to develop the second derivative test for functions of two variables. However, if time permits, some interesting applications still lie ahead. Theorem 4 is used to prove Theorem 6 in Section 7.3, which in turn is used to develop the singular value decomposition.

1. a. $\mathbf{x}^T A\mathbf{x} = [x_1 \quad x_2] \begin{bmatrix} 5 & 1/3 \\ 1/3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 5x_1^2 + \frac{2}{3}x_1x_2 + x_2^2$

b. When $\mathbf{x}=\begin{bmatrix}6\\1\end{bmatrix}$, $\mathbf{x}^T A\mathbf{x}=5(6)^2+(2/3)(6)(1)+(1)^2=185.$

c. When $\mathbf{x}=\begin{bmatrix}1\\3\end{bmatrix}$, $\mathbf{x}^T A\mathbf{x}=5(1)^2+(2/3)(1)(3)+(3)^2=16.$

2. **a**. $\mathbf{x}^T A\mathbf{x}=\begin{bmatrix}x_1 & x_2 & x_3\end{bmatrix}\begin{bmatrix}3 & 1 & 0\\1 & 1 & 2\\0 & 2 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=3x_1^2+x_2^2+2x_1x_2+4x_2x_3$

b. When $\mathbf{x}=\begin{bmatrix}-2\\-1\\5\end{bmatrix}$, $\mathbf{x}^T A\mathbf{x}=3(-2)^2+(-1)^2+2(-2)(-1)+4(-1)(5)=-3.$

c. When $\mathbf{x}=\begin{bmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix}$, $\mathbf{x}^T A\mathbf{x}=3(1/\sqrt{3})^2+(1/\sqrt{3})^2+2(1/\sqrt{3})(1/\sqrt{3})+4(1/\sqrt{3})(1/\sqrt{3})=10/3.$

3. **a**. The matrix of the quadratic form is $\begin{bmatrix}3 & -2\\-2 & 5\end{bmatrix}$.

b. The matrix of the quadratic form is $\begin{bmatrix}3 & 1\\1 & 0\end{bmatrix}$.

4. **a**. The matrix of the quadratic form is $\begin{bmatrix}5 & 8\\8 & -5\end{bmatrix}$.

b. The matrix of the quadratic form is $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$.

5. **a**. The matrix of the quadratic form is $\begin{bmatrix}3 & -3 & 4\\-3 & 2 & -2\\4 & -2 & -5\end{bmatrix}$.

b. The matrix of the quadratic form is $\begin{bmatrix}0 & 3 & 2\\3 & 0 & -5\\2 & -5 & 0\end{bmatrix}$.

6. **a**. The matrix of the quadratic form is $\begin{bmatrix}3 & 2 & -3\\2 & -2 & 0\\-3 & 0 & 5\end{bmatrix}$.

b. The matrix of the quadratic form is $\begin{bmatrix}0 & -1 & 0\\-1 & 0 & 2\\0 & 2 & 4\end{bmatrix}$.

7. The matrix of the quadratic form is $A=\begin{bmatrix}1 & 5\\ 5 & 1\end{bmatrix}$. The eigenvalues of A are 6 and –4. An eigenvector for $\lambda = 6$ is $\begin{bmatrix}1\\ 1\end{bmatrix}$, which may be normalized to $\mathbf{u}_1=\begin{bmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}$. An eigenvector for $\lambda = -4$ is $\begin{bmatrix}-1\\ 1\end{bmatrix}$, which may be normalized to $\mathbf{u}_2=\begin{bmatrix}-1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}$. Then $A=PDP^{-1}$, where $P=[\mathbf{u}_1\quad \mathbf{u}_2]=\begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$ and $D=\begin{bmatrix}6 & 0\\ 0 & -4\end{bmatrix}$. The desired change of variable is $\mathbf{x} = P\mathbf{y}$, and the new quadratic form is $\mathbf{x}^TA\mathbf{x}=(P\mathbf{y})^TA(P\mathbf{y})=\mathbf{y}^TP^TAP\mathbf{y}=\mathbf{y}^TD\mathbf{y}=6y_1^2-4y_2^2$.

8. The matrix of the quadratic form is $A=\begin{bmatrix}9 & -4 & 4\\ -4 & 7 & 0\\ 4 & 0 & 11\end{bmatrix}$. The eigenvalues of A are 3, 9, and 15. An eigenvector for $\lambda = 3$ is $\begin{bmatrix}-2\\ -2\\ 1\end{bmatrix}$, which may be normalized to $\mathbf{u}_1=\begin{bmatrix}-2/3\\ -2/3\\ 1/3\end{bmatrix}$. An eigenvector for $\lambda = 9$ is $\begin{bmatrix}-1\\ 2\\ 2\end{bmatrix}$, which may be normalized to $\mathbf{u}_2=\begin{bmatrix}-1/3\\ 2/3\\ 2/3\end{bmatrix}$. An eigenvector for $\lambda = 15$ is $\begin{bmatrix}2\\ -1\\ 2\end{bmatrix}$, which may be normalized to $\mathbf{u}_3=\begin{bmatrix}2/3\\ -1/3\\ 2/3\end{bmatrix}$. Then $A=PDP^{-1}$, where $P=[\mathbf{u}_1\quad \mathbf{u}_2\quad \mathbf{u}_3]=\begin{bmatrix}-2/3 & -1/3 & 2/3\\ -2/3 & 2/3 & -1/3\\ 1/3 & 2/3 & 2/3\end{bmatrix}$ and $D=\begin{bmatrix}3 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 15\end{bmatrix}$. The desired change of variable is $\mathbf{x} = P\mathbf{y}$, and the new quadratic form is
$\mathbf{x}^TA\mathbf{x}=(P\mathbf{y})^TA(P\mathbf{y})=\mathbf{y}^TP^TAP\mathbf{y}=\mathbf{y}^TD\mathbf{y}=3y_1^2+9y_2^2+15y_3^2$.

9. The matrix of the quadratic form is $A=\begin{bmatrix}4 & -2\\ -2 & 4\end{bmatrix}$. The eigenvalues of A are 6 and 2, so the quadratic form is positive definite. An eigenvector for $\lambda=6$ is $\begin{bmatrix}-1\\ 1\end{bmatrix}$, which may be normalized to $\mathbf{u}_1=\begin{bmatrix}-1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}$. An eigenvector for $\lambda=2$ is $\begin{bmatrix}1\\ 1\end{bmatrix}$, which may be normalized to $\mathbf{u}_2=\begin{bmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}$. Then $A=PDP^{-1}$, where $P=[\mathbf{u}_1\quad \mathbf{u}_2]=\begin{bmatrix}-1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$ and $D=\begin{bmatrix}6 & 0\\ 0 & 2\end{bmatrix}$. The desired change of variable is $\mathbf{x} = P\mathbf{y}$, and the new quadratic form is
$\mathbf{x}^TA\mathbf{x}=(P\mathbf{y})^TA(P\mathbf{y})=\mathbf{y}^TP^TAP\mathbf{y}=\mathbf{y}^TD\mathbf{y}=6y_1^2+2y_2^2$.

10. The matrix of the quadratic form is $A=\begin{bmatrix}2 & 3\\ 3 & -6\end{bmatrix}$. The eigenvalues of A are -7 and 3, so the quadratic form is indefinite. An eigenvector for $\lambda=-7$ is $\begin{bmatrix}-1\\ 3\end{bmatrix}$, which may be normalized to $\mathbf{u}_1=\begin{bmatrix}-1/\sqrt{10}\\ 3/\sqrt{10}\end{bmatrix}$. An eigenvector for $\lambda=3$ is $\begin{bmatrix}3\\ 1\end{bmatrix}$, which may be normalized to $\mathbf{u}_2=\begin{bmatrix}3/\sqrt{10}\\ 1/\sqrt{10}\end{bmatrix}$. Then $A=PDP^{-1}$, where $P=[\mathbf{u}_1 \quad \mathbf{u}_2]=\begin{bmatrix}-1/\sqrt{10} & 3/\sqrt{10}\\ 3/\sqrt{10} & 1/\sqrt{10}\end{bmatrix}$ and $D=\begin{bmatrix}-7 & 0\\ 0 & 3\end{bmatrix}$. The desired change of variable is **x** = *P***y**, and the new quadratic form is
$\mathbf{x}^TA\mathbf{x}=(P\mathbf{y})^TA(P\mathbf{y})=\mathbf{y}^TP^TAP\mathbf{y}=\mathbf{y}^TD\mathbf{y}=-7y_1^2+3y_2^2$

11. The matrix of the quadratic form is $A=\begin{bmatrix}2 & -2\\ -2 & -1\end{bmatrix}$. The eigenvalues of A are -2 and 3, so the quadratic form is indefinite. An eigenvector for $\lambda=-2$ is $\begin{bmatrix}1\\ 2\end{bmatrix}$, which may be normalized to $\mathbf{u}_1=\begin{bmatrix}1/\sqrt{5}\\ 2/\sqrt{5}\end{bmatrix}$. An eigenvector for $\lambda=3$ is $\begin{bmatrix}-2\\ 1\end{bmatrix}$, which may be normalized to $\mathbf{u}_2=\begin{bmatrix}-2/\sqrt{5}\\ 1/\sqrt{5}\end{bmatrix}$. Then $A=PDP^{-1}$, where $P=[\mathbf{u}_1 \quad \mathbf{u}_2]=\begin{bmatrix}1/\sqrt{5} & -2/\sqrt{5}\\ 2/\sqrt{5} & 1/\sqrt{5}\end{bmatrix}$ and $D=\begin{bmatrix}-2 & 0\\ 0 & 3\end{bmatrix}$. The desired change of variable is **x** = *P***y**, and the new quadratic form is
$\mathbf{x}^TA\mathbf{x}=(P\mathbf{y})^TA(P\mathbf{y})=\mathbf{y}^TP^TAP\mathbf{y}=\mathbf{y}^TD\mathbf{y}=-2y_1^2+3y_2^2$.

12. The matrix of the quadratic form is $A=\begin{bmatrix}-1 & -1\\ -1 & -1\end{bmatrix}$. The eigenvalues of A are –2 and 0, so the quadratic form is negative semidefinite. An eigenvector for $\lambda=-2$ is $\begin{bmatrix}1\\ 1\end{bmatrix}$, which may be normalized to $\mathbf{u}_1=\begin{bmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}$. An eigenvector for $\lambda=0$ is $\begin{bmatrix}-1\\ 1\end{bmatrix}$, which may be normalized to $\mathbf{u}_2=\begin{bmatrix}-1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}$. Then $A=PDP^{-1}$, where $P=[\mathbf{u}_1 \quad \mathbf{u}_2]=\begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$ and $D=\begin{bmatrix}-2 & 0\\ 0 & 0\end{bmatrix}$. The desired change of variable is **x** = *P***y**, and the new quadratic form is
$\mathbf{x}^TA\mathbf{x}=(P\mathbf{y})^TA(P\mathbf{y})=\mathbf{y}^TP^TAP\mathbf{y}=\mathbf{y}^TD\mathbf{y}=-2y_1^2$.

13. The matrix of the quadratic form is $A=\begin{bmatrix}1 & -3\\ -3 & 9\end{bmatrix}$. The eigenvalues of A are 10 and 0, so the quadratic form is positive semidefinite. An eigenvector for $\lambda=10$ is $\begin{bmatrix}1\\ -3\end{bmatrix}$, which may be normalized to $\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix}$. An eigenvector for $\lambda = 0$ is $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$, which may be normalized to $\mathbf{u}_2 = \begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$. Then $A = PDP^{-1}$, where $P = [\mathbf{u}_1 \quad \mathbf{u}_2] = \begin{bmatrix} 1/\sqrt{10} & 3/\sqrt{10} \\ -3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}$ and $D = \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix}$. The desired change of variable is $\mathbf{x} = P\mathbf{y}$, and the new quadratic form is

$$\mathbf{x}^T A\mathbf{x} = (P\mathbf{y})^T A(P\mathbf{y}) = \mathbf{y}^T P^T AP\mathbf{y} = \mathbf{y}^T D\mathbf{y} = 10y_1^2$$

14. The matrix of the quadratic form is $A = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$. The eigenvalues of A are -1 and 4, so the quadratic form is indefinite. An eigenvector for $\lambda = -1$ is $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$, which may be normalized to $\mathbf{u}_1 = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$. An eigenvector for $\lambda = 4$ is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$, which may be normalized to $\mathbf{u}_2 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$. Then $A = PDP^{-1}$, where $P = [\mathbf{u}_1 \quad \mathbf{u}_2] = \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$ and $D = \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$. The desired change of variable is $\mathbf{x} = P\mathbf{y}$, and the new quadratic form is

$$\mathbf{x}^T A\mathbf{x} = (P\mathbf{y})^T A(P\mathbf{y}) = \mathbf{y}^T P^T AP\mathbf{y} = \mathbf{y}^T D\mathbf{y} = -y_1^2 + 4y_2^2$$

15. The matrix of the quadratic form is $A = \begin{bmatrix} -3 & 2 & 2 & 2 \\ 2 & -7 & 0 & 0 \\ 2 & 0 & -10 & 3 \\ 2 & 0 & 3 & -10 \end{bmatrix}$. The eigenvalues of A are $-13,\ -9,\ -7$ and -1 so the quadratic form is negative definite. The corresponding eigenvectors may be computed: $\lambda = -1: \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \lambda = -7: \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix}, \lambda = -9: \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \lambda = -13: \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}$. These eigenvectors may be normalized to form the columns of P, and $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 3/\sqrt{12} & 0 & -1/2 & 0 \\ 1/\sqrt{12} & -2/\sqrt{6} & 1/2 & 0 \\ 1/\sqrt{12} & 1/\sqrt{6} & 1/2 & -1/\sqrt{2} \\ 1/\sqrt{12} & 1/\sqrt{6} & 1/2 & 1/\sqrt{2} \end{bmatrix} \text{and } D = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & -9 & 0 \\ 0 & 0 & 0 & -13 \end{bmatrix}.$$

The desired change of variable is $\mathbf{x} = P\mathbf{y}$, and the new quadratic form is

$$\mathbf{x}^T A\mathbf{x} = (P\mathbf{y})^T A(P\mathbf{y}) = \mathbf{y}^T P^T AP\mathbf{y} = \mathbf{y}^T D\mathbf{y} = -y_1^2 - 7y_2^2 - 9y_3^2 - 13y_4^2$$

16. The matrix of the quadratic form is $A=\begin{bmatrix}4&4&0&-3\\4&4&3&0\\0&3&4&4\\-3&0&4&4\end{bmatrix}$. The eigenvalues of A are -1 and 9, so the quadratic form is indefinite. The corresponding eigenvectors may be computed:

$\lambda=-1:\ \left\{\begin{bmatrix}4\\-5\\3\\0\end{bmatrix},\begin{bmatrix}5\\-4\\0\\3\end{bmatrix}\right\},\ \lambda=9:\ \left\{\begin{bmatrix}4\\5\\3\\0\end{bmatrix},\begin{bmatrix}-5\\-4\\0\\3\end{bmatrix}\right\}$. These bases may be converted via orthogonal projections and scalings to orthogonal bases for the respective eigenspaces:

$\lambda=-1:\ \left\{\begin{bmatrix}4\\-5\\3\\0\end{bmatrix},\begin{bmatrix}3\\0\\-4\\5\end{bmatrix}\right\},\ \lambda=9:\ \left\{\begin{bmatrix}4\\5\\3\\0\end{bmatrix},\begin{bmatrix}-3\\0\\4\\5\end{bmatrix}\right\}$. Normalize these vectors to form the columns of P, and $A=PDP^{-1}$, where $P=\begin{bmatrix}4/\sqrt{50}&3/\sqrt{50}&4/\sqrt{50}&-3/\sqrt{50}\\-5/\sqrt{50}&0&5/\sqrt{50}&0\\3/\sqrt{50}&-4/\sqrt{50}&3/\sqrt{50}&4/\sqrt{50}\\0&5/\sqrt{50}&0&5/\sqrt{50}\end{bmatrix}$, and

$D=\begin{bmatrix}-1&0&0&0\\0&-1&0&0\\0&0&9&0\\0&0&0&9\end{bmatrix}$. The desired change of variable is $\mathbf{x}=P\mathbf{y}$, and the new quadratic form is

$\mathbf{x}^TA\mathbf{x}=(P\mathbf{y})^TA(P\mathbf{y})=\mathbf{y}^TP^TAP\mathbf{y}=\mathbf{y}^TD\mathbf{y}=-1y_1^2-1y_2^2+9y_3^2+9y_4^2$.

17. The matrix of the quadratic form is $A=\begin{bmatrix}11&8&0&-6\\8&11&6&0\\0&6&11&8\\-6&0&8&11\end{bmatrix}$. The eigenvalues of A are 21 and 1, so the quadratic form is positive definite. The corresponding eigenvectors may be computed:

$\lambda=1:\ \left\{\begin{bmatrix}4\\-5\\3\\0\end{bmatrix},\begin{bmatrix}5\\-4\\0\\3\end{bmatrix}\right\},\ \lambda=21:\ \left\{\begin{bmatrix}4\\5\\3\\0\end{bmatrix},\begin{bmatrix}-5\\-4\\0\\3\end{bmatrix}\right\}$. These bases may be converted via orthogonal projections and scalings to orthogonal bases for the respective eigenspaces:

$\lambda=1:\ \left\{\begin{bmatrix}4\\-5\\3\\0\end{bmatrix},\begin{bmatrix}3\\0\\-4\\5\end{bmatrix}\right\},\ \lambda=21:\ \left\{\begin{bmatrix}4\\5\\3\\0\end{bmatrix},\begin{bmatrix}-3\\0\\4\\5\end{bmatrix}\right\}$. Normalize the vectors to form the columns of P, and $A = PDP^{-1}$, where $P = \begin{bmatrix} 4/\sqrt{50} & 3/\sqrt{50} & 4/\sqrt{50} & -3/\sqrt{50} \\ -5/\sqrt{50} & 0 & 5/\sqrt{50} & 0 \\ 3/\sqrt{50} & -4/\sqrt{50} & 3/\sqrt{50} & 4/\sqrt{50} \\ 0 & 5/\sqrt{50} & 0 & 5/\sqrt{50} \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 21 & 0 \\ 0 & 0 & 0 & 21 \end{bmatrix}$.

The desired change of variable is $\mathbf{x} = P\mathbf{y}$, and the new quadratic form is
$\mathbf{x}^T A\mathbf{x} = (P\mathbf{y})^T A(P\mathbf{y}) = \mathbf{y}^T P^T AP\mathbf{y} = \mathbf{y}^T D\mathbf{y} = y_1^2 + y_2^2 + 21y_3^2 + 21y_4^2$.

18. The matrix of the quadratic form is $A = \begin{bmatrix} 2 & -3 & -3 & -3 \\ -3 & 2 & -3 & -3 \\ -3 & -3 & 0 & -1 \\ -3 & -3 & -1 & 0 \end{bmatrix}$. The eigenvalues of A are -7, 1, and 5, so the quadratic form is indefinite. The corresponding eigenvectors may be computed:

$\lambda = -7: \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \lambda = 1: \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \lambda = 5: \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$. These eigenvectors may be normalized to

form the columns of P, and $A = PDP^{-1}$, where $P = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} & -1/2 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} & -1/2 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 & 1/2 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 1/2 \end{bmatrix}$ and

$D = \begin{bmatrix} -7 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$. The desired change of variable is $\mathbf{x} = P\mathbf{y}$, and the new quadratic form is

$\mathbf{x}^T A\mathbf{x} = (P\mathbf{y})^T A(P\mathbf{y}) = \mathbf{y}^T P^T AP\mathbf{y} = \mathbf{y}^T D\mathbf{y} = -7y_1^2 + y_2^2 + 5y_3^2 + 5y_4^2$.

19. Since 8 is larger than 5, the x_2^2 term should be as large as possible. Since $x_1^2 + x_2^2 = 1$, the largest value that x_2 can take is 1, and $x_1 = 0$ when $x_2 = 1$. Thus the largest value the quadratic form can take when $\mathbf{x}^T\mathbf{x} = 1$ is 5(0) + 8(1) = 8.

20. Since 5 is larger in absolute value than –3, the x_1^2 term should be as large as possible. Since $x_1^2 + x_2^2 = 1$, the largest value that x_1 can take is 1, and $x_2 = 0$ when $x_1 = 1$. Thus the largest value the quadratic form can take when $\mathbf{x}^T\mathbf{x} = 1$ is 5(1) – 3(0) = 5.

21. True. See the definition before Example 1, even though a nonsymmetric matrix could be used to compute values of a quadratic form.

22. False. See the paragraph before Example 1.

23. True. See the paragraph following Example 3.

24. False. The matrix P must be orthogonal and must make $P^T AP$ diagonal. See the paragraph before Example 4.

25. True. The columns of P in Theorem 4 are eigenvectors of A. See the Diagonalization Theorem in Section 5.3.

26. True. See Theorem 5(a).

27. False. $Q(\mathbf{x}) = 0$ when $\mathbf{x} = \mathbf{0}$.

28. True. See the definition before Theorem 5.

29. True. See the Numerical Note after Example 6.

30. False. See Theorem 5(b). If $\mathbf{x}^T A\mathbf{x}$ has only negative values for $\mathbf{x} \neq \mathbf{0}$, then $\mathbf{x}^T A\mathbf{x}$ is negative definite.

31. The characteristic polynomial of A may be written in two ways:

$\det(A-\lambda I) = \det\begin{bmatrix} a-\lambda & b \\ b & d-\lambda \end{bmatrix} = \lambda^2 - (a+d)\lambda + ad - b^2$ and

$(\lambda-\lambda_1)(\lambda-\lambda_2) = \lambda^2 - (\lambda_1+\lambda_2)\lambda + \lambda_1\lambda_2$. The coefficients in these polynomials may be equated to obtain $\lambda_1 + \lambda_2 = a + d$ and $\lambda_1\lambda_2 = ad - b^2 = \det A$.

32. If $\det A > 0$, then by Exercise 31, $\lambda_1\lambda_2 > 0$, so that λ_1 and λ_2 have the same sign; also, $ad = \det A + b^2 > 0$.

a. If $\det A > 0$ and $a > 0$, then $d > 0$ also, since $ad > 0$. By Exercise 31, $\lambda_1 + \lambda_2 = a + d > 0$. Since λ_1 and λ_2 have the same sign, they are both positive. So Q is positive definite by Theorem 5.

b. If $\det A > 0$ and $a < 0$, then $d < 0$ also, since $ad > 0$. By Exercise 31, $\lambda_1 + \lambda_2 = a + d < 0$. Since λ_1 and λ_2 have the same sign, they are both negative. So Q is negative definite by Theorem 5.

c. If $\det A < 0$, then by Exercise 31, $\lambda_1\lambda_2 < 0$. Thus λ_1 and λ_2 have opposite signs. So Q is indefinite by Theorem 5.

33. Exercise 34 in Section 7.1 showed that $B^T B$ is symmetric. Also $\mathbf{x}^T B^T B\mathbf{x} = (B\mathbf{x})^T B\mathbf{x} = \|B\mathbf{x}\| \geq 0$, so the quadratic form is positive semidefinite, and the matrix $B^T B$ is positive semidefinite. Suppose that B is square and invertible. Then if $\mathbf{x}^T B^T B\mathbf{x} = 0$, $\|B\mathbf{x}\| = 0$ and $B\mathbf{x} = 0$. Since B is invertible, $\mathbf{x} = \mathbf{0}$. Thus if $\mathbf{x} \neq \mathbf{0}$, $\mathbf{x}^T B^T B\mathbf{x} > 0$ and $B^T B$ is positive definite.

34. Let $A = PDP^T$, where $P^T = P^{-1}$. The eigenvalues of A are all positive: denote them $\lambda_1, \ldots, \lambda_n$. Let C be the diagonal matrix with $\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}$ on its diagonal. Then $D = C^2 = C^T C$. If $B = PCP^T$, then B is positive definite because its eigenvalues are the positive numbers on the diagonal of C. Also

$B^T B = (PCP^T)^T(PCP^T) = (P^{TT}C^T P^T)(PCP^T) = PC^T CP^T = PDP^T = A$ since $P^T P = I$.

35. Since the eigenvalues of A and B are all positive, the quadratic forms $\mathbf{x}^T A\mathbf{x}$ and $\mathbf{x}^T B\mathbf{x}$ are positive definite by Theorem 5. Let $\mathbf{x} \neq \mathbf{0}$. Then $\mathbf{x}^T A\mathbf{x} > 0$ and $\mathbf{x}^T B\mathbf{x} > 0$, so $\mathbf{x}^T(A+B)\mathbf{x} = \mathbf{x}^T A\mathbf{x} + \mathbf{x}^T B\mathbf{x} > 0$, and the quadratic form $\mathbf{x}^T(A+B)\mathbf{x}$ is positive definite. Note that $A + B$ is also a symmetric matrix. Thus by Theorem 5 all the eigenvalues of $A + B$ must be positive.

36. The eigenvalues of A are all positive by Theorem 5. Since the eigenvalues of A^{-1} are the reciprocals of the eigenvalues of A (see Exercise 33 in Section 5.1), the eigenvalues of A^{-1} are all positive. Note that A^{-1} is also a symmetric matrix. By Theorem 5, the quadratic form $\mathbf{x}^T A^{-1}\mathbf{x}$ is positive definite.

7.3 - Constrained Optimization

Notes: Theorem 6 is the main result needed in the next two sections. Theorem 7 is mentioned in Example 2 of Section 7.4. Theorem 8 is needed at the very end of Section 7.5. The economic principles in Example 6 may be familiar to students who have had a course in macroeconomics.

1. The matrix of the quadratic form on the left is $A = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 6 & -2 \\ 0 & -2 & 7 \end{bmatrix}$. The equality of the quadratic forms implies that the eigenvalues of A are 9, 6, and 3. An eigenvector may be calculated for each eigenvalue and normalized: $\lambda = 9: \begin{bmatrix} 1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$, $\lambda = 6: \begin{bmatrix} 2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$, $\lambda = 3: \begin{bmatrix} -2/3 \\ 2/3 \\ 1/3 \end{bmatrix}$. A desired change of variable is $\mathbf{x} = P\mathbf{y}$, where $P = \begin{bmatrix} 1/3 & 2/3 & -2/3 \\ 2/3 & 1/3 & 2/3 \\ -2/3 & 2/3 & 1/3 \end{bmatrix}$.

2. The matrix of the quadratic form on the left is $A = \begin{bmatrix} 3 & 3 & 1 \\ 3 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix}$. The equality of the quadratic forms implies that the eigenvalues of A are 7, 4, and 0. An eigenvector may be calculated for each eigenvalue and normalized: $\lambda = 7: \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$, $\lambda = 4: \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}$, $\lambda = 0: \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$. A desired change of variable is $\mathbf{x} = P\mathbf{y}$, where $P = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{6} & -1/\sqrt{2} \\ 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{3} & 2/\sqrt{6} & 0 \end{bmatrix}$.

3. (a) By Theorem 6, the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$ is the greatest eigenvalue λ_1 of A. By Exercise 1, $\lambda_1 = 9$.

(b) By Theorem 6, the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$ occurs at a unit eigenvector $\mathbf{u}$ corresponding to the greatest eigenvalue λ_1 of A. By Exercise 1, $\mathbf{u} = \pm\begin{bmatrix} -1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$.

(c) By Theorem 7, the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the constraints $\mathbf{x}^T\mathbf{x} = 1$ and $\mathbf{x}^T\mathbf{u} = 0$ is the second greatest eigenvalue λ_2 of A. By Exercise 1, $\lambda_2 = 6$.

4. (a) By Theorem 6, the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$ is the greatest eigenvalue λ_1 of A. By Exercise 2, $\lambda_1 = 7$.

(b) By Theorem 6, the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$ occurs at a unit eigenvector $\mathbf{u}$ corresponding to the greatest eigenvalue λ_1 of A. By Exercise 2, $\mathbf{u} = \pm\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$.

(c) By Theorem 7, the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the constraints $\mathbf{x}^T\mathbf{x} = 1$ and $\mathbf{x}^T\mathbf{u} = 0$ is the second greatest eigenvalue λ_2 of A. By Exercise 2, $\lambda_2 = 4$.

5. The matrix of the quadratic form is $A = \begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = 6$ and $\lambda_2 = -4$.

(a) By Theorem 6, the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$ is the greatest eigenvalue λ_1 of A, which is 6.

(b) By Theorem 6, the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$ occurs at a unit eigenvector $\mathbf{u}$ corresponding to the greatest eigenvalue λ_1 of A. One may compute that $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_1 = 6$, so $\mathbf{u} = \pm\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

(c) By Theorem 7, the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the constraints $\mathbf{x}^T\mathbf{x} = 1$ and $\mathbf{x}^T\mathbf{u} = 0$ is the second greatest eigenvalue λ_2 of A, which is -4.

6. The matrix of the quadratic form is $A = \begin{bmatrix} 3 & 4 \\ 4 & 9 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = 11$ and $\lambda_2 = 1$.

(a) By Theorem 6, the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$ is the greatest eigenvalue λ_1 of A, which is 11.

(b) By Theorem 6, the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$ occurs at a unit eigenvector $\mathbf{u}$ corresponding to the greatest eigenvalue λ_1 of A. One may compute that $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_1 = 11$, so $\mathbf{u} = \pm\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$.

(c) By Theorem 7, the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the constraints $\mathbf{x}^T\mathbf{x} = 1$ and $\mathbf{x}^T\mathbf{u} = 0$ is the second greatest eigenvalue λ_2 of A, which is 1.

7. The eigenvalues of the matrix of the quadratic form are $\lambda_1 = 2$, $\lambda_2 = -1$, and $\lambda_3 = -4$. By Theorem 6, the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$ occurs at a unit eigenvector $\mathbf{u}$

corresponding to the greatest eigenvalue λ_1 of A. One may compute that $\begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_1 = 2$, so $\mathbf{u} = \pm\begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$.

8. The eigenvalues of the matrix of the quadratic form are $\lambda_1 = 9$, and $\lambda_2 = -3$. By Theorem 6, the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$ occurs at a unit eigenvector $\mathbf{u}$ corresponding to the greatest eigenvalue λ_1 of A. One may compute that $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ are linearly independent eigenvectors corresponding to $\lambda_1 = 9$, so $\mathbf{u}$ can be any unit vector that is a linear combination of $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$. Alternatively, $\mathbf{u}$ can be any unit vector which is orthogonal to the eigenspace corresponding to the eigenvalue $\lambda_2 = -3$. Since multiples of $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ are eigenvectors corresponding to $\lambda_2 = -3$, $\mathbf{u}$ can be any unit vector orthogonal to $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

9. This is equivalent to finding the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$. By Theorem 6, this value is the greatest eigenvalue λ_1 of the matrix of the quadratic form. The matrix of the quadratic form is $A = \begin{bmatrix} 7 & -1 \\ -1 & 3 \end{bmatrix}$, and the eigenvalues of A are $\lambda_1 = 5+\sqrt{5}$, $\lambda_2 = 5-\sqrt{5}$. Thus the desired constrained maximum value is $\lambda_1 = 5+\sqrt{5}$.

10. This is equivalent to finding the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$. By Theorem 6, this value is the greatest eigenvalue λ_1 of the matrix of the quadratic form. The matrix of the quadratic form is $A = \begin{bmatrix} -3 & -1 \\ -1 & 5 \end{bmatrix}$, and the eigenvalues of A are $\lambda_1 = 1+\sqrt{17}$, $\lambda_2 = 1-\sqrt{17}$. Thus the desired constrained maximum value is $\lambda_1 = 1+\sqrt{17}$.

11. Since $\mathbf{x}$ is an eigenvector of A corresponding to the eigenvalue 3, $A\mathbf{x} = 3\mathbf{x}$, and $\mathbf{x}^T A\mathbf{x} = \mathbf{x}^T(3\mathbf{x}) = 3(\mathbf{x}^T\mathbf{x}) = 3\|\mathbf{x}\|^2 = 3$ since $\mathbf{x}$ is a unit vector.

12. Let $\boldsymbol{x}$ be a unit eigenvector for the eigenvalue λ. Then $x^T Ax = x^T(\lambda x) = \lambda(x^T x) = \lambda$ since $\mathbf{x}^T\mathbf{x} = 1$. So λ must satisfy $m \le \lambda \le M$.

13. If $m = M$, then let $t = (1-0)m + 0M = m$ and $\mathbf{x} = \mathbf{u}_n$. Theorem 6 shows that $\mathbf{u}_n^T A\mathbf{u}_n = m$. Now suppose that $m < M$, and let t be between m and M. Then $0 \le t - m \le M - m$ and $0 \le (t-m)/(M-m) \le 1$. Let $\alpha = (t-m)/(M-m)$, and let $\mathbf{x} = \sqrt{1-\alpha}\,\mathbf{u}_n + \sqrt{\alpha}\,\mathbf{u}_1$. The vectors $\sqrt{1-\alpha}\,\mathbf{u}_n$ and $\sqrt{\alpha}\,\mathbf{u}_1$ are orthogonal because they are eigenvectors for different eigenvalues (or one of them is $\mathbf{0}$). By the Pythagorean Theorem

$\mathbf{x}^T\mathbf{x} = \|\mathbf{x}\|^2 = \|\sqrt{1-\alpha}\mathbf{u}_n\|^2 + \|\sqrt{\alpha}\mathbf{u}_1\|^2 = |1-\alpha|\|\mathbf{u}_n\|^2 + |\alpha|\|\mathbf{u}_1\|^2 = (1-\alpha)+\alpha = 1$, since $\mathbf{u}_n$ and $\mathbf{u}_1$ are unit vectors and $0 \le \alpha \le 1$. Also, since $\mathbf{u}_n$ and $\mathbf{u}_1$ are orthogonal,

$$\begin{aligned}\mathbf{x}^T A\mathbf{x} &= (\sqrt{1-\alpha}\mathbf{u}_n + \sqrt{\alpha}\mathbf{u}_1)^T A(\sqrt{1-\alpha}\mathbf{u}_n + \sqrt{\alpha}\mathbf{u}_1) \\ &= (\sqrt{1-\alpha}\mathbf{u}_n + \sqrt{\alpha}\mathbf{u}_1)^T (m\sqrt{1-\alpha}\mathbf{u}_n + M\sqrt{\alpha}\mathbf{u}_1) \\ &= |1-\alpha|\, m\mathbf{u}_n^T\mathbf{u}_n + |\alpha|\, M\mathbf{u}_1^T\mathbf{u}_1 = (1-\alpha)m + \alpha M = t\end{aligned}$$

Thus the quadratic form $\mathbf{x}^T A\mathbf{x}$ assumes every value between m and M for a suitable unit vector $\boldsymbol{x}$.

14. The matrix of the quadratic form is $A = \begin{bmatrix} 0 & 3/2 & 5/2 & 7/2 \\ 3/2 & 0 & 7/2 & 5/2 \\ 5/2 & 7/2 & 0 & 3/2 \\ 7/2 & 5/2 & 3/2 & 0 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = 15/2$, $\lambda_2 = -1/2$, $\lambda_3 = -5/2$, and $\lambda_4 = -9/2$.

(a) By Theorem 6, the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$ is the greatest eigenvalue λ_1 of A, which is 15/2.

(b) By Theorem 6, the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$ occurs at a unit eigenvector $\mathbf{u}$ corresponding to the greatest eigenvalue λ_1 of A. One may compute that $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda_1 = 15/2$, so $\mathbf{u} = \pm\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$.

(c) By Theorem 7, the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the constraints $\mathbf{x}^T\mathbf{x} = 1$ and $\mathbf{x}^T\mathbf{u} = 0$ is the second greatest eigenvalue λ_2 of A, which is $-1/2$.

15. The matrix of the quadratic form is $A = \begin{bmatrix} 4 & -3 & -5 & -5 \\ -3 & 0 & -3 & -3 \\ -5 & -3 & 0 & -1 \\ -5 & -3 & -1 & 0 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = 9$, $\lambda_2 = 3$, $\lambda_3 = 1$, and $\lambda_4 = -9$.

(a) By Theorem 6, the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the constraint $\mathbf{x}^T\mathbf{x}=1$ is the greatest eigenvalue λ_1 of A, which is 9.

(b) By Theorem 6, the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the constraint $\mathbf{x}^T\mathbf{x}=1$ occurs at a unit eigenvector $\mathbf{u}$ corresponding to the greatest eigenvalue λ_1 of A. One may compute that $\begin{bmatrix}-2\\0\\1\\1\end{bmatrix}$ is an eigenvector corresponding to $\lambda_1=9$, so $\mathbf{u}=\pm\begin{bmatrix}-2/\sqrt{6}\\0\\1/\sqrt{6}\\1/\sqrt{6}\end{bmatrix}$.

(c) By Theorem 7, the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the constraints $\mathbf{x}^T\mathbf{x}=1$ and $\mathbf{x}^T\mathbf{u}=0$ is the second greatest eigenvalue λ_2 of A, which is 3.

16. The matrix of the quadratic form is $A=\begin{bmatrix}-6&-2&-2&-2\\-2&-10&0&0\\-2&0&-13&3\\-2&0&3&-13\end{bmatrix}$. The eigenvalues of A are $\lambda_1=-4$, $\lambda_2=-10$, $\lambda_3=-12$, and $\lambda_4=-16$.

(a) By Theorem 6, the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the constraint $\mathbf{x}^T\mathbf{x}=1$ is the greatest eigenvalue λ_1 of A, which is –4.

(b) By Theorem 6, the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the constraint $\mathbf{x}^T\mathbf{x}=1$ occurs at a unit eigenvector $\mathbf{u}$ corresponding to the greatest eigenvalue λ_1 of A. One may compute that $\begin{bmatrix}-3\\1\\1\\1\end{bmatrix}$ is an eigenvector corresponding to $\lambda_1=-4$, so $\mathbf{u}=\pm\begin{bmatrix}-3/\sqrt{12}\\1/\sqrt{12}\\1/\sqrt{12}\\1/\sqrt{12}\end{bmatrix}$.

(c) By Theorem 7, the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the constraints $\mathbf{x}^T\mathbf{x}=1$ and $\mathbf{x}^T\mathbf{u}=0$ is the second greatest eigenvalue λ_2 of A, which is –10.

17. The matrix of the quadratic form is $A=\begin{bmatrix}0&1/2&3/2&15\\1/2&0&15&3/2\\3/2&15&0&1/2\\15&3/2&1/2&0\end{bmatrix}$. The eigenvalues of A are $\lambda_1=17$, $\lambda_2=13$, $\lambda_3=-14$, and $\lambda_4=-16$.

(a) By Theorem 6, the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$ is the greatest eigenvalue λ_1 of A, which is 17.

(b) By Theorem 6, the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the constraint $\mathbf{x}^T\mathbf{x} = 1$ occurs at a unit eigenvector $\mathbf{u}$ corresponding to the greatest eigenvalue λ_1 of A. One may compute that $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ is an eigenvector corresponding to $\lambda_1 = 17$, so $\mathbf{u} = \pm\begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}$.

(c) By Theorem 7, the maximum value of $\mathbf{x}^T A\mathbf{x}$ subject to the constraints $\mathbf{x}^T\mathbf{x} = 1$ and $\mathbf{x}^T\mathbf{u} = 0$ is the second greatest eigenvalue λ_2 of A, which is 13.

7.4 - The Singular Value Decomposition

Notes: The section presents a modern topic of great importance in applications, particularly in computer calculations. An understanding of the singular value decomposition is essential for advanced work in science and engineering that requires matrix computations. Moreover, the singular value decomposition explains much about the structure of matrix transformations. The SVD does for an arbitrary matrix almost what an orthogonal decomposition does for a symmetric matrix.

1. Let $A = \begin{bmatrix}1 & 0\\0 & -3\end{bmatrix}$. Then $A^T A = \begin{bmatrix}1 & 0\\0 & 9\end{bmatrix}$, and the eigenvalues of $A^T A$ are seen to be (in decreasing order) $\lambda_1 = 9$ and $\lambda_2 = 1$. Thus the singular values of A are $\sigma_1 = \sqrt{9} = 3$ and $\sigma_2 = \sqrt{1} = 1$.

2. Let $A = \begin{bmatrix}-3 & 0\\0 & 0\end{bmatrix}$. Then $A^T A = \begin{bmatrix}9 & 0\\0 & 0\end{bmatrix}$, and the eigenvalues of $A^T A$ are (in decreasing order) $\lambda_1 = 9$ and $\lambda_2 = 0$. Thus the singular values of A are $\sigma_1 = \sqrt{9} = 3$ and $\sigma_2 = \sqrt{0} = 0$.

3. Let $A = \begin{bmatrix}2 & 3\\0 & 2\end{bmatrix}$. Then $A^T A = \begin{bmatrix}4 & 6\\6 & 13\end{bmatrix}$, and the eigenvalues of $A^T A$ are (in decreasing order) $\lambda_1 = 16$ and $\lambda_2 = 1$. Thus the singular values of A are $\sigma_1 = \sqrt{16} = 4$ and $\sigma_2 = \sqrt{1} = 1$.

4. Let $A = \begin{bmatrix}3 & 0\\8 & 3\end{bmatrix}$. Then $A^T A = \begin{bmatrix}73 & 24\\24 & 9\end{bmatrix}$, and the eigenvalues of $A^T A$ are seen to be (in decreasing order) $\lambda_1 = 81$ and $\lambda_2 = 1$. Thus the singular values of A are $\sigma_1 = \sqrt{81} = 9$ and $\sigma_2 = \sqrt{1} = 1$.

5. Let $A = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}$. Then $A^T A = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$, and the eigenvalues of $A^T A$ are seen to be (in decreasing order) $\lambda_1 = 4$ and $\lambda_2 = 0$. Associated unit eigenvectors may be computed: $\lambda_1 = 4$: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\lambda_2 = 0$: $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Thus one choice for V is $V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The singular values of A are $\sigma_1 = \sqrt{4} = 2$ and $\sigma_2 = \sqrt{0} = 0$. Thus the matrix Σ is $\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$. Next compute $\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Because $A\mathbf{v}_2 = \mathbf{0}$, the only column found for U so far is $\mathbf{u}_1$. The other column of U is found by extending $\{\mathbf{u}_1\}$ to an orthonormal basis for $\mathbb{R}^2$. An easy choice is $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Let $U = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Thus

$$A = U \Sigma V^T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

6. Let $A = \begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix}$. Then $A^T A = \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix}$, and the eigenvalues of $A^T A$ are seen to be (in decreasing order) $\lambda_1 = 9$ and $\lambda_2 = 4$. Associated unit eigenvectors may be computed: $\lambda_1 = 9$: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\lambda_2 = 4$: $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Thus one choice for V is $V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The singular values of A are $\sigma_1 = \sqrt{9} = 3$ and $\sigma_2 = \sqrt{4} = 2$. Thus the matrix Σ is $\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$. Next compute $\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \frac{1}{\sigma_2} A\mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$. Since $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a basis for $\mathbb{R}^2$, let $U = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Thus

$$A = U \Sigma V^T = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

7. Let $A = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$. Then $A^T A = \begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix}$, and the characteristic polynomial of $A^T A$ is $\lambda^2 - 13\lambda + 36 = (\lambda - 9)(\lambda - 4)$, and the eigenvalues of $A^T A$ are (in decreasing order) $\lambda_1 = 9$ and $\lambda_2 = 4$. Associated unit eigenvectors may be computed: $\lambda_1 = 9$: $\begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$, $\lambda_2 = 4$: $\begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$. Thus one choice for V is $V = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$. The singular values of A are $\sigma_1 = \sqrt{9} = 3$ and $\sigma_2 = \sqrt{4} = 2$. Thus the matrix Σ is $\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$. Next compute $\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$,

$\mathbf{u}_2 = \frac{1}{\sigma_2} A\mathbf{v}_2 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$. Since $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a basis for $\mathbb{R}^2$, let $U = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$. Thus

$A = U\,\Sigma\,V^T = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$.

8. Let $A = \begin{bmatrix} 4 & 6 \\ 0 & 4 \end{bmatrix}$. Then $A^T A = \begin{bmatrix} 16 & 24 \\ 24 & 52 \end{bmatrix}$, and the characteristic polynomial of $A^T A$ is

$\lambda^2 - 68\lambda + 256 = (\lambda - 64)(\lambda - 4)$, and the eigenvalues of $A^T A$ are (in decreasing order) $\lambda_1 = 64$ and

$\lambda_2 = 4$. Associated unit eigenvectors may be computed: $\lambda_1 = 64: \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$, $\lambda_2 = 4: \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$.

Thus one choice for V is $V = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$. The singular values of A are $\sigma_1 = \sqrt{64} = 8$ and

$\sigma_2 = \sqrt{4} = 2$. Thus the matrix Σ is $\Sigma = \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix}$. Next compute $\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$,

$\mathbf{u}_2 = \frac{1}{\sigma_2} A\mathbf{v}_2 = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$. Since $\{\mathbf{u}_1, \mathbf{u}_2\}$ is a basis for $\mathbb{R}^2$, let $U = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$. Thus

$A = U\,\Sigma\,V^T = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$.

9. Let $A = \begin{bmatrix} 3 & -3 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$. Then $A^T A = \begin{bmatrix} 10 & -8 \\ -8 & 10 \end{bmatrix}$, and the characteristic polynomial of $A^T A$ is

$\lambda^2 - 20\lambda + 36 = (\lambda - 18)(\lambda - 2)$, and the eigenvalues of $A^T A$ are (in decreasing order) $\lambda_1 = 18$ and

$\lambda_2 = 2$. Associated unit eigenvectors may be computed: $\lambda_1 = 18: \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$, $\lambda_2 = 2: \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$.

Thus one choice for V is $V = \begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$. The singular values of A are $\sigma_1 = \sqrt{18} = 3\sqrt{2}$ and

$\sigma_2 = \sqrt{2}$. Thus the matrix Σ is $\Sigma = \begin{bmatrix} 3\sqrt{2} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$. Next compute $\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$,

$\mathbf{u}_2 = \frac{1}{\sigma_2} A\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Since $\{\mathbf{u}_1, \mathbf{u}_2\}$ is not a basis for $\mathbb{R}^3$, we need a unit vector $\mathbf{u}_3$ that is orthogonal

to both $\mathbf{u}_1$ and $\mathbf{u}_2$. The vector $\mathbf{u}_3$ must satisfy the set of equations $\mathbf{u}_1^T\mathbf{x} = 0$ and $\mathbf{u}_2^T\mathbf{x} = 0$. These are equivalent to the linear equations $\begin{array}{l}-x_1+0x_2+0x_3=0\\ 0x_1+0x_2+\ x_3=0\end{array}$, so $\mathbf{x}=\begin{bmatrix}0\\1\\0\end{bmatrix}$, and $\mathbf{u}_3=\begin{bmatrix}0\\1\\0\end{bmatrix}$. Therefore let

$U=\begin{bmatrix}-1&0&0\\0&0&1\\0&1&0\end{bmatrix}$. Thus $A=U\Sigma V^T=\begin{bmatrix}-1&0&0\\0&0&1\\0&1&0\end{bmatrix}\begin{bmatrix}3\sqrt{2}&0\\0&\sqrt{2}\\0&0\end{bmatrix}\begin{bmatrix}-1/\sqrt{2}&1/\sqrt{2}\\1/\sqrt{2}&1/\sqrt{2}\end{bmatrix}$.

10. Let $A=\begin{bmatrix}7&1\\5&5\\0&0\end{bmatrix}$. Then $A^TA=\begin{bmatrix}72&32\\32&26\end{bmatrix}$, and the characteristic polynomial of A^TA is

$\lambda^2-100\lambda+900=(\lambda-90)(\lambda-10)$, and the eigenvalues of A^TA are (in decreasing order) $\lambda_1=90$ and $\lambda_2=10$. Associated unit eigenvectors may be computed:

$\lambda=90$: $\begin{bmatrix}2/\sqrt{5}\\1/\sqrt{5}\end{bmatrix}$, $\lambda=10$: $\begin{bmatrix}-1/\sqrt{5}\\2/\sqrt{5}\end{bmatrix}$. Thus one choice for V is $V=\begin{bmatrix}2/\sqrt{5}&-1/\sqrt{5}\\1/\sqrt{5}&2/\sqrt{5}\end{bmatrix}$. The singular values of A are $\sigma_1=\sqrt{90}=3\sqrt{10}$ and $\sigma_2=\sqrt{10}$. Thus the matrix Σ is $\Sigma=\begin{bmatrix}3\sqrt{10}&0\\0&\sqrt{10}\\0&0\end{bmatrix}$.

Next compute $\mathbf{u}_1=\frac{1}{\sigma_1}A\mathbf{v}_1=\begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix}$, $\mathbf{u}_2=\frac{1}{\sigma_2}A\mathbf{v}_2=\begin{bmatrix}-1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix}$. Since $\{\mathbf{u}_1,\mathbf{u}_2\}$ is not a basis for $\mathbb{R}^3$, we need a unit vector $\mathbf{u}_3$ that is orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$. The vector $\mathbf{u}_3$ must satisfy the set of equations $\mathbf{u}_1^T\mathbf{x}=0$ and $\mathbf{u}_2^T\mathbf{x}=0$. These are equivalent to the linear equations

$\begin{array}{l}-\frac{1}{\sqrt{2}}x_1+\frac{1}{\sqrt{2}}x_2+0x_3=0\\ \frac{1}{\sqrt{2}}x_1+\frac{1}{\sqrt{2}}x_2+0x_3=0\end{array}$, so $\mathbf{x}=\begin{bmatrix}0\\0\\1\end{bmatrix}$, and $\mathbf{u}_3=\begin{bmatrix}0\\0\\1\end{bmatrix}$ Therefore let $U=\begin{bmatrix}1/\sqrt{2}&-1/\sqrt{2}&0\\1/\sqrt{2}&1/\sqrt{2}&0\\0&0&1\end{bmatrix}$. Thus

$$A=U\Sigma V^T=\begin{bmatrix}1/\sqrt{2}&-1/\sqrt{2}&0\\1/\sqrt{2}&1/\sqrt{2}&0\\0&0&1\end{bmatrix}\begin{bmatrix}3\sqrt{10}&0\\0&\sqrt{10}\\0&0\end{bmatrix}\begin{bmatrix}2/\sqrt{5}&1/\sqrt{5}\\-1/\sqrt{5}&2/\sqrt{5}\end{bmatrix}.$$

11. Let $A=\begin{bmatrix}-3&1\\6&-2\\6&-2\end{bmatrix}$. Then $A^TA=\begin{bmatrix}81&-27\\-27&9\end{bmatrix}$, and the characteristic polynomial of A^TA is

$\lambda^2-90\lambda=\lambda(\lambda-90)$, and the eigenvalues of A^TA are (in decreasing order) $\lambda_1=90$ and $\lambda_2=0$. Associated unit eigenvectors may be computed: $\lambda_1=90$: $\begin{bmatrix}3/\sqrt{10}\\-1/\sqrt{10}\end{bmatrix}$, $\lambda_2=0$: $\begin{bmatrix}1/\sqrt{10}\\3/\sqrt{10}\end{bmatrix}$. Thus one choice for V is $V=\begin{bmatrix} 3/\sqrt{10} & 1/\sqrt{10} \\ -1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$. The singular values of A are $\sigma_1=\sqrt{90}=3\sqrt{10}$ and $\sigma_2=\sqrt{0}=0.$ Thus the matrix Σ is $\Sigma=\begin{bmatrix} 3\sqrt{10} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$. Next compute $\mathbf{u}_1=\dfrac{1}{\sigma_1}A\mathbf{v}_1=\begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$. Because $A\mathbf{v}_2=\mathbf{0}$, the only column found for U so far is $\mathbf{u}_1$. The other columns of U can be found by extending $\{\mathbf{u}_1\}$ to an orthonormal basis for $\mathbb{R}^3$. In this case, we need two orthogonal unit vectors $\mathbf{u}_2$ and $\mathbf{u}_3$ that are orthogonal to $\mathbf{u}_1$. Each vector must satisfy the equation $\mathbf{u}_1^T\mathbf{x}=0,$ which is equivalent to the equation $-x_1+2x_2+2x_3=0.$ An orthonormal basis for the solution set of this equation is $\mathbf{u}_2=\begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix}, \mathbf{u}_3=\begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix}$. Therefore, let $U=\begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix}$. Thus

$$A=U\Sigma V^T=\begin{bmatrix} -1/3 & 2/3 & 2/3 \\ 2/3 & -1/3 & 2/3 \\ 2/3 & 2/3 & -1/3 \end{bmatrix}\begin{bmatrix} 3\sqrt{10} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3/\sqrt{10} & -1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}.$$

12. Let $A=\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$. Then $A^TA=\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, and the eigenvalues of A^TA are seen to be (in decreasing order) $\lambda_1=3$ and $\lambda_2=2$. Associated unit eigenvectors may be computed: $\lambda_1=3$: $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\lambda_2=2$: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Thus one choice for V is $V=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The singular values of A are $\sigma_1=\sqrt{3}$ and $\sigma_2=\sqrt{2}.$ Thus the matrix Σ is $\Sigma=\begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{bmatrix}$. Next compute $\mathbf{u}_1=\dfrac{1}{\sigma_1}A\mathbf{v}_1=\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$, $\mathbf{u}_2=\dfrac{1}{\sigma_2}A\mathbf{v}_2=\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$. Since $\{\mathbf{u}_1,\mathbf{u}_2\}$ is not a basis for $\mathbb{R}^3$, we need a unit vector $\mathbf{u}_3$ that is orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$. The vector $\mathbf{u}_3$ must satisfy the set of equations $\mathbf{u}_1^T\mathbf{x}=0$ and $\mathbf{u}_2^T\mathbf{x}=0.$ These are equivalent to the linear equations $\begin{matrix} x_1+x_2+x_3=0 \\ x_1+0x_2-x_3=0 \end{matrix}$, so $\mathbf{x}=\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, and $\mathbf{u}_3=\begin{bmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$.

Therefore let $U=\begin{bmatrix}1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6}\\ 1/\sqrt{3} & 0 & -2/\sqrt{6}\\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6}\end{bmatrix}$. Thus

$$A=U\Sigma V^T=\begin{bmatrix}1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6}\\ 1/\sqrt{3} & 0 & -2/\sqrt{6}\\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6}\end{bmatrix}\begin{bmatrix}\sqrt{3} & 0\\ 0 & \sqrt{2}\\ 0 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}.$$

13. Let $A=\begin{bmatrix}3 & 2 & 2\\ 2 & 3 & -2\end{bmatrix}$. Then $A^T=\begin{bmatrix}3 & 2\\ 2 & 3\\ 2 & -2\end{bmatrix}$, $A^{TT}A^T=AA^T=\begin{bmatrix}17 & 8\\ 8 & 17\end{bmatrix}$, and the eigenvalues of $A^{TT}A^T$ are seen to be (in decreasing order) $\lambda_1=25$ and $\lambda_2=9$. Associated unit eigenvectors may be computed: $\lambda=25$: $\begin{bmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}$, $\lambda=9$: $\begin{bmatrix}-1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}$. Thus one choice for V is $V=\begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$. The singular values of A^T are $\sigma_1=\sqrt{25}=5$ and $\sigma_2=\sqrt{9}=3$. Thus the matrix Σ is $\Sigma=\begin{bmatrix}5 & 0\\ 0 & 3\\ 0 & 0\end{bmatrix}$. Next compute $\mathbf{u}_1=\frac{1}{\sigma_1}A^T\mathbf{v}_1=\begin{bmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\\ 0\end{bmatrix}$, $\mathbf{u}_2=\frac{1}{\sigma_2}A^T\mathbf{v}_2=\begin{bmatrix}-1/\sqrt{18}\\ 1/\sqrt{18}\\ -4/\sqrt{18}\end{bmatrix}$. Since $\{\mathbf{u}_1,\mathbf{u}_2\}$ is not a basis for $\mathbb{R}^3$, we need a unit vector $\mathbf{u}_3$ that is orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$. The vector $\mathbf{u}_3$ must satisfy the set of equations $\mathbf{u}_1^T\mathbf{x}=0$ and $\mathbf{u}_2^T\mathbf{x}=0$. These are equivalent to the linear equations $\begin{aligned}x_1+x_2+0x_3&=0\\ -x_1+x_2-4x_3&=0\end{aligned}$, so $\mathbf{x}=\begin{bmatrix}-2\\ 2\\ 1\end{bmatrix}$, and $\mathbf{u}_3=\begin{bmatrix}-2/3\\ 2/3\\ 1/3\end{bmatrix}$. Therefore let

$U=\begin{bmatrix}1/\sqrt{2} & -1/\sqrt{18} & -2/3\\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3\\ 0 & -4/\sqrt{18} & 1/3\end{bmatrix}$. Thus

$A^T=U\Sigma V^T=\begin{bmatrix}1/\sqrt{2} & -1/\sqrt{18} & -2/3\\ 1/\sqrt{2} & 1/\sqrt{18} & 2/3\\ 0 & -4/\sqrt{18} & 1/3\end{bmatrix}\begin{bmatrix}5 & 0\\ 0 & 3\\ 0 & 0\end{bmatrix}\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ -1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$. An SVD for A is computed by taking transposes: $A=\begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}5 & 0 & 0\\ 0 & 3 & 0\end{bmatrix}\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2} & 0\\ -1/\sqrt{18} & 1/\sqrt{18} & -4/\sqrt{18}\\ -2/3 & 2/3 & 1/3\end{bmatrix}$.

14. From Exercise 7, $A = U\Sigma V^T$ with $V = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$. Since the first column of V is unit eigenvector associated with the greatest eigenvalue λ_1 of $A^T A$, so the first column of V is a unit vector at which $\|A\mathbf{x}\|$ is maximized.

15. **a**. Since A has 2 nonzero singular values, rank A = 2.

b. By Example 6, $\{\mathbf{u}_1, \mathbf{u}_2\} = \left\{ \begin{bmatrix} .40 \\ .37 \\ -.84 \end{bmatrix}, \begin{bmatrix} -.78 \\ -.33 \\ -.52 \end{bmatrix} \right\}$ is a basis for Col A and $\{\mathbf{v}_3\} = \left\{ \begin{bmatrix} .58 \\ -.58 \\ .58 \end{bmatrix} \right\}$ is a basis for Nul A.

16. **a**. Since A has 2 nonzero singular values, rank A = 2.

b. By Example 6, $\{\mathbf{u}_1, \mathbf{u}_2\} = \left\{ \begin{bmatrix} -.86 \\ .31 \\ .41 \end{bmatrix}, \begin{bmatrix} -.11 \\ .68 \\ -.73 \end{bmatrix} \right\}$ is a basis for Col A and

$\{\mathbf{v}_3, \mathbf{v}_4\} = \left\{ \begin{bmatrix} .65 \\ .08 \\ -.16 \\ -.73 \end{bmatrix}, \begin{bmatrix} -.34 \\ .42 \\ -.84 \\ -.08 \end{bmatrix} \right\}$ is a basis for Nul A.

17. First note that the determinant of an orthogonal matrix is ±1, because $1 = \det I = \det U^T U = (\det U^T)(\det U) = (\det U)^2$. Suppose that A is square and $A = U\Sigma V^T$. Then Σ is square, and $\det A = (\det U)(\det \Sigma)(\det V^T) = \pm\det\Sigma = \pm\sigma_1 \ldots \sigma_n$.

18. Let $A = U\Sigma V^T = U\Sigma V^{-1}$. Since A is square and invertible, rank A = n, and all of the entries on the diagonal of Σ must be nonzero. So $A^{-1} = (U\Sigma V^{-1})^{-1} = V\Sigma^{-1}U^{-1} = V\Sigma^{-1}U^T$.

19. Since U and V are orthogonal matrices,

$$A^T A = (U\Sigma V^T)^T U\Sigma V^T = V\Sigma^T U^T U\Sigma V^T = V(\Sigma^T\Sigma)V^T = V(\Sigma^T\Sigma)V^{-1}$$

If $\sigma_1, \ldots, \sigma_r$ are the diagonal entries in Σ, then $\Sigma^T\Sigma$ is a diagonal matrix with diagonal entries $\sigma_1^2, \ldots, \sigma_r^2$ and possibly some zeros. Thus V diagonalizes $A^T A$ and the columns of V are eigenvectors of $A^T A$ by the Diagonalization Theorem in Section 5.3. Likewise

$$AA^T = U\Sigma V^T (U\Sigma V^T)^T = U\Sigma V^T V\Sigma^T U^T = U(\Sigma\Sigma^T)U^T = U(\Sigma\Sigma^T)U^{-1}$$

so U diagonalizes AA^T and the columns of U must be eigenvectors of AA^T. Moreover, the Diagonalization Theorem states that $\sigma_1^2, \ldots, \sigma_r^2$ are the nonzero eigenvalues of $A^T A$. Hence $\sigma_1, \ldots, \sigma_r$ are the nonzero singular values of A.

20. Let $A = U\Sigma V^T$. The matrix PU is orthogonal, because P and U are both orthogonal. (See Exercise 37 in Section 6.2). So the equation $PA = (PU)\Sigma V^T$ has the form required for a singular value decomposition. By Exercise 19, the diagonal entries in Σ are the singular values of PA.

21. The right singular vector $\mathbf{v}_1$ is an eigenvector for the largest eigenvector λ_1 of $A^T A$. By Theorem 7 in Section 7.3, the second largest eigenvalue λ_2 is the maximum of $\mathbf{x}^T(A^T A)\mathbf{x}$ over all unit vectors orthogonal to $\mathbf{v}_1$. Since $\mathbf{x}^T(A^T A)\mathbf{x} = \| A\mathbf{x} \|^2$, the square root of λ_2, which is the second largest singular value of A, is the maximum of $\| A\mathbf{x} \|$ over all unit vectors orthogonal to $\mathbf{v}_1$.

22. If A is positive definite, then $A = PDP^T$, where P is an orthogonal matrix and D is a diagonal matrix. The diagonal entries of D are positive because they are the eigenvalues of a positive definite matrix. Since P is an orthogonal matrix, $PP^T = I$ and the square matrix P^T is invertible. Moreover, $(P^T)^{-1} = (P^{-1})^{-1} = P = (P^T)^T$, so P^T is an orthogonal matrix. Thus the factorization $A = PDP^T$ has the properties that make it a singular value decomposition.

23. From the proof of Theorem 10, $U\Sigma = [\sigma_1\mathbf{u}_1 \quad \ldots \quad \sigma_r\mathbf{u}_r \quad \mathbf{0} \quad \ldots \quad \mathbf{0}]$. The column-row expansion of the product $(U\Sigma)V^T$ shows that $A = (U\Sigma)V^T = (U\Sigma)\begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix} = \sigma_1\mathbf{u}_1\mathbf{v}_1^T + \ldots + \sigma_r\mathbf{u}_r\mathbf{v}_r^T$, where r is the rank of A.

24. From Exercise 23, $A^T = \sigma_1\mathbf{v}_1\mathbf{u}_1^T + \ldots + \sigma_r\mathbf{v}_r\mathbf{u}_r^T$. Then since $\mathbf{u}_i^T\mathbf{u}_j = \begin{cases} 0 & \text{for } i \neq j \\ 1 & \text{for } i = j \end{cases}$,

$$A^T\mathbf{u}_j = (\sigma_1\mathbf{v}_1\mathbf{u}_1^T + \ldots + \sigma_r\mathbf{v}_r\mathbf{u}_r^T)\mathbf{u}_j = (\sigma_j\mathbf{v}_j\mathbf{u}_j^T)\mathbf{u}_j = \sigma_j\mathbf{v}_j(\mathbf{u}_j^T\mathbf{u}_j) = \sigma_j\mathbf{v}_j$$

25. Consider the SVD for the standard matrix A of T, say $A = U\Sigma V^T$. Let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $C = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ be bases for $\mathbb{R}^n$ and $\mathbb{R}^m$ constructed respectively from the columns of V and U. Since the columns of V are orthogonal, $V^T\mathbf{v}_j = \mathbf{e}_j$, where $\mathbf{e}_j$ is the jth column of the $n \times n$ identity matrix. To find the matrix of T relative to B and C, compute $T(\mathbf{v}_j) = A\mathbf{v}_j = U\Sigma V^T\mathbf{v}_j = U\Sigma\mathbf{e}_j = U\sigma_j\mathbf{e}_j = \sigma_j U\mathbf{e}_j = \sigma_j\mathbf{u}_j$, so $[T(\mathbf{v}_j)]_C = \sigma_j\mathbf{e}_j$. Formula (4) in the discussion at the beginning of Section 5.4 shows that the "diagonal" matrix Σ is the matrix of T relative to B and C.

26. Let $A = \begin{bmatrix} -18 & 13 & -4 & 4 \\ 2 & 19 & -4 & 12 \\ -14 & 11 & -12 & 8 \\ -2 & 21 & 4 & 8 \end{bmatrix}$. Then $A^T A = \begin{bmatrix} 528 & -392 & 224 & -176 \\ -392 & 1092 & -176 & 536 \\ 224 & -176 & 192 & -128 \\ -176 & 536 & -128 & 288 \end{bmatrix}$, and the eigenvalues of $A^T A$ are found to be (in decreasing order) $\lambda_1 = 1600$, $\lambda_2 = 400$, $\lambda_3 = 100$, and $\lambda_4 = 0$.

Associated unit eigenvectors may be computed: $\lambda_1 : \begin{bmatrix} -.4 \\ .8 \\ -.2 \\ .4 \end{bmatrix}$, $\lambda_2 : \begin{bmatrix} .8 \\ .4 \\ .4 \\ .2 \end{bmatrix}$, $\lambda_3 : \begin{bmatrix} .4 \\ -.2 \\ -.8 \\ .4 \end{bmatrix}$, $\lambda_4 : \begin{bmatrix} -.2 \\ -.4 \\ .4 \\ .8 \end{bmatrix}$.

Thus one choice for V is $V = \begin{bmatrix} -.4 & .8 & .4 & -.2 \\ .8 & .4 & -.2 & -.4 \\ -.2 & .4 & -.8 & .4 \\ .4 & .2 & .4 & .8 \end{bmatrix}$. The singular values of A are $\sigma_1 = 40$,

$\sigma_1 = 20$, $\sigma_3 = 10$, and $\sigma_4 = 0$. Thus the matrix Σ is $\Sigma = \begin{bmatrix} 40 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Next compute

$\mathbf{u}_1 = \dfrac{1}{\sigma_1} A\mathbf{v}_1 = \begin{bmatrix} .5 \\ .5 \\ .5 \\ .5 \end{bmatrix}$, $\mathbf{u}_2 = \dfrac{1}{\sigma_2} A\mathbf{v}_2 = \begin{bmatrix} -.5 \\ .5 \\ -.5 \\ .5 \end{bmatrix}$, $\mathbf{u}_3 = \dfrac{1}{\sigma_3} A\mathbf{v}_3 = \begin{bmatrix} -.5 \\ .5 \\ .5 \\ -.5 \end{bmatrix}$. Because $A\mathbf{v}_4 = \mathbf{0}$, only three columns of U have been found so far. The last column of U can be found by extending $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ to an orthonormal basis for $\mathbb{R}^4$. The vector $\mathbf{u}_4$ must satisfy the set of equations $\mathbf{u}_1^T\mathbf{x} = 0$, $\mathbf{u}_2^T\mathbf{x} = 0$, and $\mathbf{u}_3^T\mathbf{x} = 0$. These are equivalent to the linear equations $\begin{matrix} x_1 + x_2 + x_3 + x_4 = 0 \\ -x_1 + x_2 - x_3 + x_4 = 0, \\ -x_1 + x_2 + x_3 - x_4 = 0 \end{matrix}$ so $\mathbf{x} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$,

and $\mathbf{u}_4 = \begin{bmatrix} -.5 \\ -.5 \\ .5 \\ .5 \end{bmatrix}$. Therefore, let $U = \begin{bmatrix} .5 & -.5 & -.5 & -.5 \\ .5 & .5 & .5 & -.5 \\ .5 & -.5 & .5 & .5 \\ .5 & .5 & -.5 & .5 \end{bmatrix}$. Thus

$$A = U\Sigma V^T = \begin{bmatrix} .5 & -.5 & -.5 & -.5 \\ .5 & .5 & .5 & -.5 \\ .5 & -.5 & .5 & .5 \\ .5 & .5 & -.5 & .5 \end{bmatrix} \begin{bmatrix} 40 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -.4 & .8 & -.2 & .4 \\ .8 & .4 & .4 & .2 \\ .4 & -.2 & -.8 & .4 \\ -.2 & -.4 & .4 & .8 \end{bmatrix}.$$

27. Let $A = \begin{bmatrix} 6 & -8 & -4 & 5 & -4 \\ 2 & 7 & -5 & -6 & 4 \\ 0 & -1 & -8 & 2 & 2 \\ -1 & -2 & 4 & 4 & -8 \end{bmatrix}$. Then $A^TA = \begin{bmatrix} 41 & -32 & -38 & 14 & -8 \\ -32 & 118 & -3 & -92 & 74 \\ -38 & -3 & 121 & 10 & -52 \\ 14 & -92 & 10 & 81 & -72 \\ -8 & 74 & -52 & -72 & 100 \end{bmatrix}$, and the eigenvalues of A^TA are found to be (in decreasing order) $\lambda_1 = 270.87$, $\lambda_2 = 147.85$, $\lambda_3 = 23.73$, $\lambda_4 = 18.55$, and $\lambda_5 = 0$. Associated unit eigenvectors may be computed:

$\lambda_1: \begin{bmatrix} -.10 \\ .61 \\ -.21 \\ -.52 \\ .55 \end{bmatrix}, \lambda_2: \begin{bmatrix} -.39 \\ .29 \\ .84 \\ -.14 \\ -.19 \end{bmatrix}, \lambda_3: \begin{bmatrix} -.74 \\ -.27 \\ -.07 \\ .38 \\ .49 \end{bmatrix}, \lambda_4: \begin{bmatrix} .41 \\ -.50 \\ .45 \\ -.23 \\ .58 \end{bmatrix}, \lambda_5: \begin{bmatrix} -.36 \\ -.48 \\ -.19 \\ -.72 \\ -.29 \end{bmatrix}$. Thus one choice for V is

$V = \begin{bmatrix} -.10 & -.39 & -.74 & .41 & -.36 \\ .61 & .29 & -.27 & -.50 & -.48 \\ -.21 & .84 & -.07 & .45 & -.19 \\ -.52 & -.14 & .38 & -.23 & -.72 \\ .55 & -.19 & .49 & .58 & -.29 \end{bmatrix}$. The nonzero singular values of A are $\sigma_1 = 16.46$,

$\sigma_1 = 12.16$, $\sigma_3 = 4.87$, and $\sigma_4 = 4.31$. Thus the matrix Σ is $\Sigma = \begin{bmatrix} 16.46 & 0 & 0 & 0 & 0 \\ 0 & 12.16 & 0 & 0 & 0 \\ 0 & 0 & 4.87 & 0 & 0 \\ 0 & 0 & 0 & 4.31 & 0 \end{bmatrix}$.

Next compute $\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1 = \begin{bmatrix} -.57 \\ .63 \\ .07 \\ -.51 \end{bmatrix}, \mathbf{u}_2 = \frac{1}{\sigma_2} A\mathbf{v}_2 = \begin{bmatrix} -.65 \\ -.24 \\ -.63 \\ .34 \end{bmatrix}$, $\mathbf{u}_3 = \frac{1}{\sigma_3} A\mathbf{v}_3 = \begin{bmatrix} -.42 \\ -.68 \\ .53 \\ -.29 \end{bmatrix}$,

$\mathbf{u}_4 = \frac{1}{\sigma_4} A\mathbf{v}_4 = \begin{bmatrix} .27 \\ -.29 \\ -.56 \\ -.73 \end{bmatrix}$. Since $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ is a basis for $\mathbb{R}^4$, let $U = \begin{bmatrix} -.57 & -.65 & -.42 & .27 \\ .63 & -.24 & -.68 & -.29 \\ .07 & -.63 & .53 & -.56 \\ -.51 & .34 & -.29 & -.73 \end{bmatrix}$.

Thus $A = U\Sigma V^T$

$$= \begin{bmatrix} -.57 & -.65 & -.42 & .27 \\ .63 & -.24 & -.68 & -.29 \\ .07 & -.63 & .53 & -.56 \\ -.51 & .34 & -.29 & -.73 \end{bmatrix} \begin{bmatrix} 16.46 & 0 & 0 & 0 & 0 \\ 0 & 12.16 & 0 & 0 & 0 \\ 0 & 0 & 4.87 & 0 & 0 \\ 0 & 0 & 0 & 4.31 & 0 \end{bmatrix} \begin{bmatrix} -.10 & .61 & -.21 & -.52 & .55 \\ -.39 & .29 & .84 & -.14 & -.19 \\ -.74 & -.27 & -.07 & .38 & .49 \\ .41 & -.50 & .45 & -.23 & .58 \\ -.36 & -.48 & -.19 & -.72 & -.29 \end{bmatrix}$$

.

28. Let $A = \begin{bmatrix} 4 & 0 & -7 & -7 \\ -6 & 1 & 11 & 9 \\ 7 & -5 & 10 & 19 \\ -1 & 2 & 3 & -1 \end{bmatrix}$. Then $A^T A = \begin{bmatrix} 102 & -43 & -27 & 52 \\ -43 & 30 & -33 & -88 \\ -27 & -33 & 279 & 335 \\ 52 & -16 & 335 & 492 \end{bmatrix}$, and the eigenvalues of

$A^T A$ are found to be (in decreasing order) $\lambda_1 = 749.979$, $\lambda_2 = 146.201$, $\lambda_3 = 6.82061$, and $\lambda_4 = .000001$ The singular values of A are thus $\sigma_1 = 27.3857$, $\sigma_2 = 12.0914$, $\sigma_3 = 2.61163$, and $\sigma_4 = .001156$ The condition number $\sigma_1 / \sigma_4 \approx 23683$

29. Let $A = \begin{bmatrix} 5 & 3 & 1 & 7 & 9 \\ 6 & 4 & 2 & 8 & -8 \\ 7 & 5 & 3 & 10 & 9 \\ 9 & 6 & 4 & -9 & -5 \\ 8 & 5 & 2 & 11 & 4 \end{bmatrix}$. Then $A^T A = \begin{bmatrix} 255 & 168 & 90 & 160 & 47 \\ 168 & 111 & 60 & 104 & 30 \\ 90 & 60 & 34 & 39 & 8 \\ 160 & 104 & 39 & 415 & 178 \\ 47 & 30 & 8 & 178 & 267 \end{bmatrix}$, and the eigenvalues of $A^T A$ are found to be (in decreasing order) $\lambda_1 = 672.589$, $\lambda_2 = 280.745$, $\lambda_3 = 127.503$, $\lambda_4 = 1.163$, and $\lambda_5 = 1.428\times10^{-7}$. The singular values of A are thus $\sigma_1 = 25.9343$, $\sigma_2 = 16.7554$, $\sigma_3 = 11.2917$, $\sigma_4 = 1.07853$, and $\sigma_5 = .000377928$. The condition number $\sigma_1 / \sigma_5 = 68,622$.

7.5 - Applications to Image Processing and Statistics

Notes: The application presented here has turned out to be of interest to a wide variety of students, including engineers. I cover this in Course Syllabus 3 described in the front mater of the text, but I only have time to mention the idea briefly to my other classes.

1. The matrix of observations is $X = \begin{bmatrix} 19 & 22 & 6 & 3 & 2 & 20 \\ 12 & 6 & 9 & 15 & 13 & 5 \end{bmatrix}$ and the sample mean is $M = \frac{1}{6}\begin{bmatrix} 72 \\ 60 \end{bmatrix} = \begin{bmatrix} 12 \\ 10 \end{bmatrix}$. The mean-deviation form B is obtained by subtracting M from each column of X, so $B = \begin{bmatrix} 7 & 10 & -6 & -9 & -10 & 8 \\ 2 & -4 & -1 & 5 & 3 & -5 \end{bmatrix}$. The sample covariance matrix is

$$S = \frac{1}{6-1}BB^T = \frac{1}{5}\begin{bmatrix} 430 & -135 \\ -135 & 80 \end{bmatrix} = \begin{bmatrix} 86 & -27 \\ -27 & 16 \end{bmatrix}$$

2. The matrix of observations is $X = \begin{bmatrix} 1 & 5 & 2 & 6 & 7 & 3 \\ 3 & 11 & 6 & 8 & 15 & 11 \end{bmatrix}$ and the sample mean is $M = \frac{1}{6}\begin{bmatrix} 24 \\ 54 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$. The mean-deviation form B is obtained by subtracting M from each column of X, so $B = \begin{bmatrix} -3 & 1 & -2 & 2 & 3 & -1 \\ -6 & 2 & -3 & -1 & 6 & 2 \end{bmatrix}$. The sample covariance matrix is

$$S = \frac{1}{6-1}BB^T = \frac{1}{5}\begin{bmatrix} 28 & 40 \\ 40 & 90 \end{bmatrix} = \begin{bmatrix} 5.6 & 8 \\ 8 & 18 \end{bmatrix}.$$

3. The principal components of the data are the unit eigenvectors of the sample covariance matrix S. One computes that (in descending order) the eigenvalues of $S = \begin{bmatrix} 86 & -27 \\ -27 & 16 \end{bmatrix}$ are $\lambda_1 = 95.2041$ and $\lambda_2 = 6.79593$. One further computes that corresponding eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} -2.93348 \\ 1 \end{bmatrix}$ and

$\mathbf{v}_2 = \begin{bmatrix} .340892 \\ 1 \end{bmatrix}$. These vectors may be normalized to find the principal components, which are $\mathbf{u}_1 = \begin{bmatrix} .946515 \\ -.322659 \end{bmatrix}$ for $\lambda_1 = 95.2041$ and $\mathbf{u}_2 = \begin{bmatrix} .322659 \\ .946515 \end{bmatrix}$ for $\lambda_2 = 6.79593$.

4. The principal components of the data are the unit eigenvectors of the sample covariance matrix *S*. One computes that (in descending order) the eigenvalues of $S = \begin{bmatrix} 5.6 & 8 \\ 8 & 18 \end{bmatrix}$ are $\lambda_1 = 21.9213$ and $\lambda_2 = 1.67874$. One further computes that corresponding eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} .490158 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -2.04016 \\ 1 \end{bmatrix}$. These vectors may be normalized to find the principal components, which are $\mathbf{u}_1 = \begin{bmatrix} .44013 \\ .897934 \end{bmatrix}$ for $\lambda_1 = 21.9213$ and $\mathbf{u}_2 = \begin{bmatrix} -.897934 \\ .44013 \end{bmatrix}$ for $\lambda_2 = 1.67874$.

5. The largest eigenvalue of $S = \begin{bmatrix} 164.12 & 32.73 & 81.04 \\ 32.73 & 539.44 & 249.13 \\ 81.04 & 249.13 & 189.11 \end{bmatrix}$ is $\lambda_1 = 677.497$, and the first principal component of the data is the unit eigenvector corresponding to λ_1, which is $\mathbf{u}_1 = \begin{bmatrix} .129554 \\ .874423 \\ .467547 \end{bmatrix}$. The fraction of the total variance that is contained in this component is $\lambda_1 / \mathrm{tr}(S) = 677.497/(164.12 + 539.44 + 189.11) = .758956$ so 75.8956% of the variance of the data is contained in the first principal component.

6. The largest eigenvalue of $S = \begin{bmatrix} 29.64 & 18.38 & 5.00 \\ 18.38 & 20.82 & 14.06 \\ 5.00 & 14.06 & 29.21 \end{bmatrix}$ is $\lambda_1 = 51.6957$, and the first principal component of the data is the unit eigenvector corresponding to λ_1, which is $\mathbf{u}_1 = \begin{bmatrix} .615525 \\ .599424 \\ .511683 \end{bmatrix}$. Thus one choice for the new variable is $y_1 = .615525x_1 + .599424x_2 + .511683x_3$. The fraction of the total variance that is contained in this component is $\lambda_1 / \mathrm{tr}(S) = 51.6957/(29.64 + 20.82 + 29.21) = .648872$, so 64.8872% of the variance of the data is explained by y_1.

7. Since the unit eigenvector corresponding to $\lambda_1 = 95.2041$ is $\mathbf{u}_1 = \begin{bmatrix} .946515 \\ -.322659 \end{bmatrix}$, one choice for the new variable is $y_1 = .946515x_1 - .322659x_2$. The fraction of the total variance that is contained in this component is $\lambda_1 / \mathrm{tr}(S) = 95.2041/(86 + 16) = .933374$, so 93.3374% of the variance of the data is explained by y_1.

8. Since the unit eigenvector corresponding to $\lambda_1 = 21.9213$ is $\mathbf{u}_1 = \begin{bmatrix} .44013 \\ .897934 \end{bmatrix}$, one choice for the new variable is $y_1 = .44013x_1 + .897934x_2$. The fraction of the total variance that is contained in this component is $\lambda_1 / tr(S) = 21.9213/(5.6+18) = .928869$, so 92.8869% of the variance of the data is explained by y_1.

9. The largest eigenvalue of $S = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix}$ is $\lambda_1 = 9$, and the first principal component of the data is the unit eigenvector corresponding to λ_1, which is $\mathbf{u}_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$. Thus one choice for y is $y = (1/3)x_1 + (2/3)x_2 + (2/3)x_3$, and the variance of y is $\lambda_1 = 9$.

10. The largest eigenvalue of $S = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 11 & 4 \\ 2 & 4 & 5 \end{bmatrix}$ is $\lambda_1 = 15$, and the first principal component of the data is the unit eigenvector corresponding to λ_1, which is $\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$. Thus one choice for y is $y = (1/\sqrt{6})x_1 + (2/\sqrt{6})x_2 + (1/\sqrt{6})x_3$, and the variance of y is $\lambda_1 = 15$.

11. **a**. If $\mathbf{w}$ is the vector in $\mathbb{R}^N$ with a 1 in each position, then $[\mathbf{X}_1 \ \ \dots \ \ \mathbf{X}_N]\mathbf{w} = \mathbf{X}_1 + \dots + \mathbf{X}_N = \mathbf{0}$ since the $\mathbf{X}_k$ are in mean-deviation form. Then

$$[\mathbf{Y}_1 \ \ \dots \ \ \mathbf{Y}_N]\mathbf{w} = [P^T\mathbf{X}_1 \ \ \dots \ \ P^T\mathbf{X}_N]\mathbf{w} = P^T[\mathbf{X}_1 \ \ \dots \ \ \mathbf{X}_N]\mathbf{w} = P^T\mathbf{0} = \mathbf{0}$$

Thus $\mathbf{Y}_1 + \dots + \mathbf{Y}_N = \mathbf{0}$, and the $\mathbf{Y}_k$ are in mean-deviation form.

b. By part a., the covariance matrix $S_{\mathbf{Y}}$ of $\mathbf{Y}_1, \dots, \mathbf{Y}_N$ is

$$\begin{aligned} S_{\mathbf{Y}} &= \frac{1}{N-1}[\mathbf{Y}_1 \ \ \dots \ \ \mathbf{Y}_N][\mathbf{Y}_1 \ \ \dots \ \ \mathbf{Y}_N]^T \\ &= \frac{1}{N-1}P^T[\mathbf{X}_1 \ \ \dots \ \ \mathbf{X}_N](P^T[\mathbf{X}_1 \ \ \dots \ \ \mathbf{X}_N])^T \\ &= P^T\left(\frac{1}{N-1}[\mathbf{X}_1 \ \ \dots \ \ \mathbf{X}_N][\mathbf{X}_1 \ \ \dots \ \ \mathbf{X}_N]^T\right)P = P^TSP \end{aligned}$$

since the $\mathbf{X}_k$ are in mean-deviation form.

12. By Exercise 11, the change of variables $\mathbf{X} = P\mathbf{Y}$ changes the covariance matrix S of X into the covariance matrix P^TSP of $\mathbf{Y}$. The total variance of the data as described by $\mathbf{Y}$ is $\text{tr}(P^TSP)$. However, since P^TSP is similar to S, they have the same trace (by Exercise 27 in Section 5.4). Thus the total variance of the data is unchanged by the change of variables $\mathbf{X} = P\mathbf{Y}$.

13. Let $\mathbf{M}$ be the sample mean for the data, and let $\hat{\mathbf{X}}_k = \mathbf{X}_k - \mathbf{M}$. Let $B = \begin{bmatrix} \hat{\mathbf{X}}_1 & \dots & \hat{\mathbf{X}}_N \end{bmatrix}$ be the matrix of observations in mean-deviation form. By the row-column expansion of BB^T, the sample covariance matrix is

$$\begin{aligned} S &= \frac{1}{N-1} BB^T \\ &= \frac{1}{N-1} \begin{bmatrix} \hat{\mathbf{X}}_1 & \dots & \hat{\mathbf{X}}_N \end{bmatrix} \begin{bmatrix} \hat{\mathbf{X}}_1^T \\ \vdots \\ \hat{\mathbf{X}}_N^T \end{bmatrix} \\ &= \frac{1}{N-1} \sum_{k=1}^{N} \hat{\mathbf{X}}_k \hat{\mathbf{X}}_k^T = \frac{1}{N-1} \sum_{k=1}^{N} (\mathbf{X}_k - \mathbf{M})(\mathbf{X}_k - \mathbf{M})^T \end{aligned}$$

Chapter 7 - Supplementary Exercises

1. True. This is just part of Theorem 2 in Section 7.1. The proof appears just before the statement of the theorem.

2. False. A counterexample is $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

3. True. This is proved in the first part of the proof of Theorem 6 in Section 7.3. It is also a consequence of Theorem 7 in Section 6.2.

4. False. The principal axes of $\mathbf{x}^T A\mathbf{x}$ are the columns of any *orthogonal* matrix P that diagonalizes A. *Note*: When A has an eigenvalue whose eigenspace has dimension greater than 1, the principal axes are not uniquely determined.

5. False. A counterexample is $P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. The columns here are orthogonal but not orthonormal.

6. False. See Example 6 in Section 7.2.

7. False. A counterexample is $A = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then $\mathbf{x}^T A\mathbf{x} = 2 > 0$, but $\mathbf{x}^T A\mathbf{x}$ is an indefinite quadratic form.

8. True. This is basically the Principal Axes Theorem from Section 7.2. Any quadratic form can be written as $\mathbf{x}^T A\mathbf{x}$ for some symmetric matrix A.

9. False. See Example 3 in Section 7.3.

10. False. The maximum value must be computed over the set of *unit* vectors. Without a restriction on the norm of $\mathbf{x}$, the values of $\mathbf{x}^T A\mathbf{x}$ can be made as large as desired.

11. False. Any orthogonal change of variable $\mathbf{x} = P\mathbf{y}$ changes a positive definite quadratic form into another positive definite quadratic form. Proof: By Theorem 5 of Section 7.2., the classification of a quadratic form is determined by the eigenvalues of the matrix of the form. Given a form $\mathbf{x}^T A\mathbf{x}$, the matrix of the new quadratic form is $P^{-1}AP$, which is similar to A and thus has the same eigenvalues as A.

12. False. The term "definite eigenvalue" is undefined and therefore meaningless.

13. True. If $\mathbf{x} = P\mathbf{u}$, then $\mathbf{x}^T A\mathbf{x} = (P\mathbf{u})^T A(P\mathbf{u}) = \mathbf{u}^T P^T AP\mathbf{u} = \mathbf{u}^T P^{-1}AP\mathbf{u}$.

14. False. A counterexample is $U = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$. The columns of U must be *orthonormal* to make $UU^T\mathbf{x}$ the orthogonal projection of $\mathbf{x}$ onto Col U.

15. True. This follows from the discussion in Example 2 of Section 7.4., which refers to a proof given in Example 1.

16. True. Theorem 10 in Section 7.4 writes the decomposition in the form $U\Sigma V^T$, where U and V are orthogonal matrices. In this case, V^T is also an orthogonal matrix. Proof: Since V is orthogonal, V is invertible and $V^{-1} = V^T$. Then $(V^T)^{-1} = (V^{-1})^T = (V^T)^T$, and since V is square and invertible, V^T is an orthogonal matrix.

17. False. A counterexample is $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$. The singular values of A are 2 and 1, but the singular values of $A^T A$ are 4 and 1.

18. **a**. Each term in the expansion of A is symmetric by Exercise 41 in Section 7.1. The fact that $(B+C)^T = B^T + C^T$ implies that any sum of symmetric matrices is symmetric, so A is symmetric.

b. Since $\mathbf{u}_1^T\mathbf{u}_1 = 1$ and $\mathbf{u}_j^T\mathbf{u}_1 = 0$ for $j \neq 1$,

$$A\mathbf{u}_1 = (\lambda_1\mathbf{u}_1\mathbf{u}_1^T)\mathbf{u}_1 + \ldots + (\lambda_n\mathbf{u}_n\mathbf{u}_n^T)\mathbf{u}_1 = \lambda_1\mathbf{u}_1(\mathbf{u}_1^T\mathbf{u}_1) + \ldots + \lambda_n\mathbf{u}_n(\mathbf{u}_n^T\mathbf{u}_1) = \lambda_1\mathbf{u}_1$$

Since $\mathbf{u}_1 \neq \mathbf{0}$, λ_1 is an eigenvalue of A. A similar argument shows that λ_j is an eigenvalue of A for $j = 2, \ldots, n$.

19. If rank $A = r$, then dimNul $A = n - r$ by the Rank Theorem. So 0 is an eigenvalue of A with multiplicity $n - r$, and of the n terms in the spectral decomposition of A exactly $n - r$ are zero. The remaining r terms (which correspond to nonzero eigenvalues) are all rank 1 matrices, as mentioned in the discussion of the spectral decomposition.

20. **a**. By Theorem 3 in Section 6.1, $(\text{Col } A)^\perp = \text{Nul } A^T = \text{Nul } A$ since $A^T = A$.

b. Let $\mathbf{y}$ be in $\mathbb{R}^n$. By the Orthogonal Decomposition Theorem in Section 6.3, $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$, where $\hat{\mathbf{y}}$ is in Col A and $\mathbf{z}$ is in $(\text{Col } A)^\perp$. By part a., $\mathbf{z}$ is in Nul A.

21. If $A\mathbf{v} = \lambda\mathbf{v}$ for some nonzero λ, then $\mathbf{v} = \lambda^{-1}A\mathbf{v} = A(\lambda^{-1}\mathbf{v})$, which shows that $\mathbf{v}$ is a linear combination of the columns of A.

22. Because A is symmetric, there is an orthonormal eigenvector basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ for $\mathbb{R}^n$. Let $r = \text{rank } A$. If $r = 0$, then $A = O$ and the decomposition of Exercise 20(b) is $\mathbf{y} = \mathbf{0} + \mathbf{y}$ for each $\mathbf{y}$ in $\mathbb{R}^n$; if $r = n$ then the decomposition is $\mathbf{y} = \mathbf{y} + \mathbf{0}$ for each $\mathbf{y}$ in $\mathbb{R}^n$.

Assume that $0 < r < n$. Then $\dim \text{Nul } A = n - r$ by the Rank Theorem, and so 0 is an eigenvalue of A with multiplicity $n - r$. Hence there are r nonzero eigenvalues, counted according to their multiplicities. Renumber the eigenvector basis if necessary so that $\mathbf{u}_1, \ldots, \mathbf{u}_r$ are the eigenvectors corresponding to the nonzero eigenvalues. By Exercise 21, $\mathbf{u}_1, \ldots, \mathbf{u}_r$ are in Col A. Also, $\mathbf{u}_{r+1}, \ldots, \mathbf{u}_n$ are in Nul A because these vectors are eigenvectors corresponding to the eigenvalue 0. For $\mathbf{y}$ in $\mathbb{R}^n$, there are scalars $c_1, \ldots, c_n$ such that

$$\mathbf{y} = \underbrace{c_1\mathbf{u}_1 + \ldots + c_r\mathbf{u}_r}_{\hat{\mathbf{y}}} + \underbrace{c_{r+1}\mathbf{u}_{r+1} + \ldots + c_n\mathbf{u}_n}_{\mathbf{z}}$$

This provides the decomposition in Exercise 20(b).

23. If $A = R^T R$ and R is invertible, then A is positive definite by Exercise 33 in Section 7.2.

Conversely, suppose that A is positive definite. Then by Exercise 34 in Section 7.2, $A = B^T B$ for some positive definite matrix B. Since the eigenvalues of B are positive, 0 is not an eigenvalue of B and B is invertible. Thus the columns of B are linearly independent. By Theorem 12 in Section 6.4, $B = QR$ for some $n \times n$ matrix Q with orthonormal columns and some upper triangular matrix R with positive entries on its diagonal. Since Q is a square matrix, $Q^T Q = I$, and

$$A = B^T B = (QR)^T (QR) = R^T Q^T QR = R^T R$$

and R has the required properties.

24. Suppose that A is positive definite, and consider a Cholesky factorization of $A = R^T R$ with R upper triangular and having positive entries on its diagonal. Let D be the diagonal matrix whose diagonal entries are the entries on the diagonal of R. Since right-multiplication by a diagonal matrix scales the columns of the matrix on its left, the matrix $L = R^T D^{-1}$ is lower triangular with 1's on its diagonal. If $U = DR$, then $A = R^T D^{-1} DR = LU$.

25. If A is an $m \times n$ matrix and $\mathbf{x}$ is in $\mathbb{R}^n$, then $\mathbf{x}^T A^T A\mathbf{x} = (A\mathbf{x})^T (A\mathbf{x}) = \|A\mathbf{x}\|^2 \geq 0$. Thus $A^T A$ is positive semidefinite. By Exercise 30 in Section 6.5, $\text{rank } A^T A = \text{rank } A$.

26. If rank $G = r$, then $\dim \text{Nul } G = n - r$ by the Rank Theorem. Hence 0 is an eigenvalue of G with multiplicity $n - r$, and the spectral decomposition of G is

$$G = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \ldots + \lambda_r \mathbf{u}_r \mathbf{u}_r^T$$

Also $\lambda_1, \ldots, \lambda_r$ are positive because G is positive semidefinite. Thus

$$G = \left(\sqrt{\lambda_1}\mathbf{u}_1\right)\left(\sqrt{\lambda_1}\mathbf{u}_1^{\,T}\right) + \ldots + \left(\sqrt{\lambda_r}\mathbf{u}_r\right)\left(\sqrt{\lambda_r}\mathbf{u}_r^{\,T}\right)$$

By the column-row expansion of a matrix product, $G = BB^T$ where B is the $n \times r$ matrix $B = \left[\sqrt{\lambda_1}\mathbf{u}_1 \quad \dots \quad \sqrt{\lambda_r}\mathbf{u}_r\right]$. Finally, $G = A^T A$ for $A = B^T$.

27. Let $A = U\Sigma V^T$ be a singular value decomposition of A. Since U is orthogonal, $U^T U = I$ and $A = U\Sigma U^T U V^T = PQ$ where $P = U\Sigma U^T = U\Sigma U^{-1}$ and $Q = UV^T$. Since Σ is symmetric, P is symmetric, and P has nonnegative eigenvalues because it is similar to Σ, which is diagonal with nonnegative diagonal entries. Thus P is positive semidefinite. The matrix Q is orthogonal since it is the product of orthogonal matrices.

28. **a**. Because the columns of V_r are orthonormal,

$$AA^+\mathbf{y} = (U_r D V_r^T)(V_r D^{-1} U_r^T)\mathbf{y} = (U_r D D^{-1} U_r^T)\mathbf{y} = U_r U_r^T \mathbf{y}$$

Since $U_r U_r^T \mathbf{y}$ is the orthogonal projection of $\mathbf{y}$ onto $\text{Col}\, U_r$ by Theorem 10 in Section 6.3, and since $\text{Col}\, U_r = \text{Col}\, A$ by (5) in Example 6 of Section 7.4, $AA^+\mathbf{y}$ is the orthogonal projection of $\mathbf{y}$ onto Col A.

b. Because the columns of U_r are orthonormal,

$$A^+ A\mathbf{x} = (V_r D^{-1} U_r^T)(U_r D V_r^T)\mathbf{x} = (V_r D^{-1} D V_r^T)\mathbf{x} = V_r V_r^T \mathbf{x}$$

Since $V_r V_r^T \mathbf{x}$ is the orthogonal projection of $\mathbf{x}$ onto $\text{Col}\, V_r$ by Theorem 10 in Section 6.3, and since $\text{Col}\, V_r = \text{Row}\, A$ by (8) in Example 6 of Section 7.4, $A^+ A\mathbf{x}$ is the orthogonal projection of $\mathbf{x}$ onto Row A.

c. Using the reduced singular value decomposition, the definition of A^+, and the associativity of matrix multiplication gives:

$$\begin{aligned}
AA^+A &= (U_r D V_r^T)(V_r D^{-1} U_r^T)(U_r D V_r^T) = (U_r D D^{-1} U_r^T)(U_r D V_r^T) \\
&= U_r D D^{-1} D V_r^T = U_r D V_r^T = A \\
A^+ A A^+ &= (V_r D^{-1} U_r^T)(U_r D V_r^T)(V_r D^{-1} U_r^T) = (V_r D^{-1} D V_r^T)(V_r D^{-1} U_r^T) \\
&= V_r D^{-1} D D^{-1} U_r^T = V_r D^{-1} U_r^T = A^+
\end{aligned}$$

29. **a**. If $\mathbf{b} = A\mathbf{x}$, then $\mathbf{x}^+ = A^+\mathbf{b} = A^+ A\mathbf{x}$. By Exercise 28(a), $\mathbf{x}^+$ is the orthogonal projection of $\mathbf{x}$ onto Row A.

b. From part (a) and Exercise 28(c), $A\mathbf{x}^+ = A(A^+ A\mathbf{x}) = (AA^+ A)\mathbf{x} = A\mathbf{x} = \mathbf{b}$.

c. Let $A\mathbf{u} = \mathbf{b}$. Since $\mathbf{x}^+$ is the orthogonal projection of $\mathbf{x}$ onto Row A, the Pythagorean Theorem shows that $\|\mathbf{u}\|^2 = \|\mathbf{x}^+\|^2 + \|\mathbf{u} - \mathbf{x}^+\|^2 \geq \|\mathbf{x}^+\|^2$, with equality only if $\mathbf{u} = \mathbf{x}^+$.

30. The least-squares solutions of $A\mathbf{x} = \mathbf{b}$ are precisely the solutions of $A\mathbf{x} = \hat{\mathbf{b}}$, where $\hat{\mathbf{b}}$ is the orthogonal projection of $\mathbf{b}$ onto Col A. From Exercise 29, the minimum length solution of $A\mathbf{x} = \hat{\mathbf{b}}$ is $A^+\hat{\mathbf{b}}$, so $A^+\hat{\mathbf{b}}$ is the minimum length least-squares solution of $A\mathbf{x} = \mathbf{b}$. However, $\hat{\mathbf{b}} = AA^+\mathbf{b}$ by Exercise 28(a) and hence $A^+\hat{\mathbf{b}} = A^+ AA^+\mathbf{b} = \mathrm{A}^+\mathbf{b}$ by Exercise 28(c). Thus $A^+\mathbf{b}$ is the minimum length least-squares solution of $A\mathbf{x} = \mathbf{b}$.

31. The reduced SVD of A is $A = U_r D V_r^T$, where

$$U_r = \begin{bmatrix} .966641 & .253758 & -.034804 \\ .185205 & -.786338 & -.589382 \\ .125107 & -.398296 & .570709 \\ .125107 & -.398296 & .570709 \end{bmatrix}, D = \begin{bmatrix} 9.84443 & 0 & 0 \\ 0 & 2.62466 & 0 \\ 0 & 0 & 1.09467 \end{bmatrix},$$

$$\text{and } V_r = \begin{bmatrix} -.313388 & .009549 & .633795 \\ -.313388 & .009549 & .633795 \\ -.633380 & .023005 & -.313529 \\ .633380 & -.023005 & .313529 \\ .035148 & .999379 & .002322 \end{bmatrix}$$

So the pseudoinverse $A^+ = V_r D^{-1} U_r^T$ may be calculated, as well as the solution $\hat{\mathbf{x}} = A^+\mathbf{b}$ for the system $A\mathbf{x} = \mathbf{b}$:

$$A^+ = \begin{bmatrix} -.05 & -.35 & .325 & .325 \\ -.05 & -.35 & .325 & .325 \\ -.05 & .15 & -.175 & -.175 \\ .05 & -.15 & .175 & .175 \\ .10 & -.30 & -.150 & -.150 \end{bmatrix}, \hat{\mathbf{x}} = \begin{bmatrix} .7 \\ .7 \\ -.8 \\ .8 \\ .6 \end{bmatrix}$$

Row reducing the augmented matrix for the system $A^T\mathbf{z} = \hat{\mathbf{x}}$ shows that this system has a solution, so $\hat{\mathbf{x}}$ is in $\text{Col } A^T = \text{Row } A$. A basis for Nul A is $\{\mathbf{a}_1, \mathbf{a}_2\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$, and an arbitrary element of Nul A is $\mathbf{u} = c\mathbf{a}_1 + d\mathbf{a}_2$. One computes that $\|\hat{\mathbf{x}}\| = \sqrt{131/50}$, while $\|\hat{\mathbf{x}} + \mathbf{u}\| = \sqrt{(131/50) + 2c^2 + 2d^2}$. Thus if $\mathbf{u} \neq \mathbf{0}$, $\|\hat{\mathbf{x}}\| < \|\hat{\mathbf{x}} + \mathbf{u}\|$, which confirms that $\hat{\mathbf{x}}$ is the minimum length solution to $A\mathbf{x} = \mathbf{b}$.

32. The reduced SVD of A is $A = U_r D V_r^T$, where

$$U_r = \begin{bmatrix} -.337977 & .936307 & .095396 \\ .591763 & .290230 & -.752053 \\ -.231428 & -.062526 & -.206232 \\ -.694283 & -.187578 & -.618696 \end{bmatrix}, D = \begin{bmatrix} 12.9536 & 0 & 0 \\ 0 & 1.44553 & 0 \\ 0 & 0 & .337763 \end{bmatrix},$$

$$\text{and } V_r = \begin{bmatrix} -.690099 & .721920 & .050939 \\ 0 & 0 & 0 \\ .341800 & .387156 & -.856320 \\ .637916 & .573534 & .513928 \\ 0 & 0 & 0 \end{bmatrix}$$

So the pseudoinverse $A^{+} = V_r D^{-1} U_r^T$ may be calculated, as well as the solution $\hat{\mathbf{x}} = A^{+}\mathbf{b}$ for the system $A\mathbf{x} = \mathbf{b}$:

$$A^{+} = \begin{bmatrix} .5 & 0 & -.05 & -.15 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & .5 & 1.5 \\ .5 & -1 & -.35 & -1.05 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \hat{\mathbf{x}} = \begin{bmatrix} 2.3 \\ 0 \\ 5.0 \\ -.9 \\ 0 \end{bmatrix}$$

Row reducing the augmented matrix for the system $A^T\mathbf{z} = \hat{\mathbf{x}}$ shows that this system has a solution, so $\hat{\mathbf{x}}$ is in $\text{Col } A^T = \text{Row } A$. A basis for Nul A is $\{\mathbf{a}_1, \mathbf{a}_2\} = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$, and an arbitrary element of Nul A is $\mathbf{u} = c\mathbf{a}_1 + d\mathbf{a}_2$. One computes that $\|\hat{\mathbf{x}}\| = \sqrt{311/10}$, while $\|\hat{\mathbf{x}} + \mathbf{u}\| = \sqrt{(311/10) + c^2 + d^2}$. Thus if $\mathbf{u} \neq \mathbf{0}$, $\|\hat{\mathbf{x}}\| < \|\hat{\mathbf{x}} + \mathbf{u}\|$, which confirms that $\hat{\mathbf{x}}$ is the minimum length solution to $A\mathbf{x} = \mathbf{b}$.