8 The Geometry of Vector Spaces

8.1 - Affine Combinations

Notes. This section introduces a special kinds of linear combination used to describe the sets created when a subspace is shifted away from the origin. An affine combination is a linear combination in which the coefficients sum to one. Theorems 1, 3, and 4 connect affine combinations directly to linear combinations. There are several approaches to solving many of the exercises in this section, and some of the alternatives are presented here.

1. $\mathbf{v}_1 = \begin{bmatrix}1\\2\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}-2\\2\end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix}0\\4\end{bmatrix}$4, $\mathbf{v}_4 = \begin{bmatrix}3\\7\end{bmatrix}, \mathbf{y} = \begin{bmatrix}5\\3\end{bmatrix}$

$$\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix}-3\\0\end{bmatrix}, \mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix}-1\\2\end{bmatrix}, \mathbf{v}_4 - \mathbf{v}_1 = \begin{bmatrix}2\\5\end{bmatrix}, \mathbf{y} - \mathbf{v}_1 = \begin{bmatrix}4\\1\end{bmatrix}$$

Solve $c_2(\mathbf{v}_2 - \mathbf{v}_1) + c_3(\mathbf{v}_3 - \mathbf{v}_1) + c_4(\mathbf{v}_4 - \mathbf{v}_1) = \mathbf{y} - \mathbf{v}_1$ by row reducing the augmented matrix.

$$\begin{bmatrix}-3 & -1 & 2 & 4\\0 & 2 & 5 & 1\end{bmatrix} \sim \begin{bmatrix}-3 & -1 & 2 & 4\\0 & 1 & 2.5 & .5\end{bmatrix} \sim \begin{bmatrix}-3 & 0 & 4.5 & 4.5\\0 & 1 & 2.5 & .5\end{bmatrix} \sim \begin{bmatrix}1 & 0 & -1.5 & -1.5\\0 & 1 & 2.5 & .5\end{bmatrix}$$

The general solution is $c_2 = 1.5c_4 - 1.5$, $c_3 = -2.5c_4 + .5$, with c_4 free. When $c_4 = 0$,

$\mathbf{y} - \mathbf{v}_1 = -1.5(\mathbf{v}_2 - \mathbf{v}_1) + .5(\mathbf{v}_3 - \mathbf{v}_1)$ and $\mathbf{y} = 2\mathbf{v}_1 - 1.5\mathbf{v}_2 + .5\mathbf{v}_3$

If $c_4 = 1$, then $c_2 = 0$ and

$\mathbf{y} - \mathbf{v}_1 = -2(\mathbf{v}_3 - \mathbf{v}_1) + 1(\mathbf{v}_4 - \mathbf{v}_1)$ and $\mathbf{y} = 2\mathbf{v}_1 - 2\mathbf{v}_3 + \mathbf{v}_4$

If $c_4 = 3$, then

$$\mathbf{y} - \mathbf{v}_1 = 3(\mathbf{v}_2 - \mathbf{v}_1) - 7(\mathbf{v}_3 - \mathbf{v}_1) + 3(\mathbf{v}_4 - \mathbf{v}_1) \quad \text{and} \quad \mathbf{y} = 2\mathbf{v}_1 + 3\mathbf{v}_2 - 7\mathbf{v}_3 + 3\mathbf{v}_4$$

Of course, many other answers are possible. Note that in all cases, the weights in the linear combination sum to one.

2. $\mathbf{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}-1\\2\end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix}3\\2\end{bmatrix}, \mathbf{y} = \begin{bmatrix}5\\7\end{bmatrix}$, so $\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix}-2\\1\end{bmatrix}, \mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix}2\\1\end{bmatrix}$, and $\mathbf{y} - \mathbf{v}_1 = \begin{bmatrix}4\\6\end{bmatrix}$

Solve $c_2(\mathbf{v}_2 - \mathbf{v}_1) + c_3(\mathbf{v}_3 - \mathbf{v}_1) = \mathbf{y} - \mathbf{v}_1$ by row reducing the augmented matrix:

$$\begin{bmatrix}-2 & 2 & 4\\1 & 1 & 6\end{bmatrix} \sim \begin{bmatrix}1 & -1 & -2\\0 & 2 & 8\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 2\\0 & 1 & 4\end{bmatrix}$$

The general solution is $c_2 = 2$ and $c_3 = 4$, so $\mathbf{y} - \mathbf{v}_1 = 2(\mathbf{v}_2 - \mathbf{v}_1) + 4(\mathbf{v}_3 - \mathbf{v}_1)$ and $\mathbf{y} = -5\mathbf{v}_1 + 2\mathbf{v}_2 + 4\mathbf{v}_3$. The weights sum to one, so this is an affine sum.

3. Row reduce the augmented matrix $[\mathbf{v}_2 - \mathbf{v}_1 \;\; \mathbf{v}_3 - \mathbf{v}_1 \;\; \mathbf{y} - \mathbf{v}_1]$ to find a solution for writing $\mathbf{y} - \mathbf{v}_1$ in terms of $\mathbf{v}_2 - \mathbf{v}_1$ and $\mathbf{v}_3 - \mathbf{v}_1$. Then solve for $\mathbf{y}$ to get $\mathbf{y} = -3\mathbf{v}_1 + 2\mathbf{v}_2 + 2\mathbf{v}_3$. The weights sum to one, so this is an affine sum.

4. Row reduce the augmented matrix $[\mathbf{v}_2 - \mathbf{v}_1 \;\; \mathbf{v}_3 - \mathbf{v}_1 \;\; \mathbf{y} - \mathbf{v}_1]$ to find a solution for writing $\mathbf{y} - \mathbf{v}_1$ in terms of $\mathbf{v}_2 - \mathbf{v}_1$ and $\mathbf{v}_3 - \mathbf{v}_1$. Then solve for $\mathbf{y}$ to get $\mathbf{y} = 2.6\mathbf{v}_1 - .4\mathbf{v}_2 - 1.2\mathbf{v}_3$. The weights sum to one, so this is an affine sum.

5. Since $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is an orthogonal basis, use Theorem 5 from Section 6.2 to write

$$\mathbf{p}_j = \frac{\mathbf{p}_j \cdot \mathbf{b}_1}{\mathbf{b}_1 \cdot \mathbf{b}_1}\mathbf{b}_1 + \frac{\mathbf{p}_j \cdot \mathbf{b}_2}{\mathbf{b}_2 \cdot \mathbf{b}_2}\mathbf{b}_2 + \frac{\mathbf{p}_j \cdot \mathbf{b}_3}{\mathbf{b}_3 \cdot \mathbf{b}_3}\mathbf{b}_3$$

a. $\mathbf{p}_1 = 3\mathbf{b}_1 - \mathbf{b}_2 - \mathbf{b}_3 \in \operatorname{aff} S$ since the coefficients sum to one.
b. $\mathbf{p}_2 = 2\mathbf{b}_1 + 0\mathbf{b}_2 + \mathbf{b}_3 \notin \operatorname{aff} S$ since the coefficients do not sum to one.
c. $\mathbf{p}_3 = -\mathbf{b}_1 + 2\mathbf{b}_2 + 0\mathbf{b}_3 \in \operatorname{aff} S$ since the coefficients sum to one.

6. Since $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is an orthogonal basis, use Theorem 5 from Section 6.2 to write

$$\mathbf{p}_j = \frac{\mathbf{p}_j \cdot \mathbf{b}_1}{\mathbf{b}_1 \cdot \mathbf{b}_1}\mathbf{b}_1 + \frac{\mathbf{p}_j \cdot \mathbf{b}_2}{\mathbf{b}_2 \cdot \mathbf{b}_2}\mathbf{b}_2 + \frac{\mathbf{p}_j \cdot \mathbf{b}_3}{\mathbf{b}_3 \cdot \mathbf{b}_3}\mathbf{b}_3$$

a. $\mathbf{p}_1 = -4\mathbf{b}_1 + 2\mathbf{b}_2 + 3\mathbf{b}_3 \in \operatorname{aff} S$ since the coefficients sum to one.
b. $\mathbf{p}_2 = .2\mathbf{b}_1 + .5\mathbf{b}_2 + .3\mathbf{b}_3 \in \operatorname{aff} S$ since the coefficients sum to one.
c. $\mathbf{p}_3 = \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3 \notin \operatorname{aff} S$ since the coefficients do not sum to one.

7. The matrix $[\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \mathbf{v}_3 \;\; \mathbf{p}_1 \;\; \mathbf{p}_2 \;\; \mathbf{p}_3]$ row reduces to $\begin{bmatrix} 1 & 0 & 0 & 2 & 2 & 2 \\ 0 & 1 & 0 & 1 & -4 & 2 \\ 0 & 0 & 1 & -1 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 & -5 \end{bmatrix}$.

Parts **a.**, **b.**, and **c.** use columns 4, 5, and 6, respectively, as the “augmented” column.

a. $\mathbf{p}_1 = 2\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3$, so $\mathbf{p}_1$ is in Span S. The weights do not sum to one, so $\mathbf{p}_1 \notin \operatorname{aff} S$.

b. $\mathbf{p}_2 = 2\mathbf{v}_1 - 4\mathbf{v}_2 + 3\mathbf{v}_3$, so $\mathbf{p}_2$ is in Span S. The weights sum to one, so $\mathbf{p}_2 \in \operatorname{aff} S$.

c. $\mathbf{p}_3 \notin$ Span S because $0 \neq -5$, so $\mathbf{p}_3$ cannot possibly be in aff S.

8. The matrix $[\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \mathbf{v}_3 \;\; \mathbf{p}_1 \;\; \mathbf{p}_2 \;\; \mathbf{p}_3]$ row reduces to $\begin{bmatrix} 1 & 0 & 0 & 3 & 0 & -2 \\ 0 & 1 & 0 & -1 & 0 & 6 \\ 0 & 0 & 1 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$.

Parts **a.**, **b.**, and **c.** use columns 4, 5, and 6, respectively, as the “augmented” column.

a. $\mathbf{p}_1 = 3\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3$, so $\mathbf{p}_1$ is in Span S. The weights do not sum to one, so $\mathbf{p}_1 \notin \operatorname{aff} S$.

b. $\mathbf{p}_2 \notin$ Span S because $0 \neq 1$ (column 5 is the augmented column), so $\mathbf{p}_2$ cannot possibly be in aff S.

c. $\mathbf{p}_3 = -2\mathbf{v}_1 + 6\mathbf{v}_2 - 3\mathbf{v}_3$, so $\mathbf{p}_3$ is in Span S. The weights sum to one, so $\mathbf{p}_3 \in \operatorname{aff} S$.

9. Choose $\mathbf{v}_1$ and $\mathbf{v}_2$ to be any two points on the line $\mathbf{x} = x_3\mathbf{u} + \mathbf{p}$. For example, take $x_3 = 0$ and $x_3 = 1$ to get $\mathbf{v}_1 = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ respectively. Other answers are possible.

10. Choose $\mathbf{v}_1$ and $\mathbf{v}_2$ to be any two points on the line $\mathbf{x} = x_3\mathbf{u} + \mathbf{p}$. For example, take $x_3 = 0$ and $x_3 = 1$ to get $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 6 \\ -2 \\ 2 \end{bmatrix}$ respectively. Other answers are possible.

11. True. See the definition at the beginning of this section.

12. False. If $S = \{\mathbf{x}\}$, then aff $S = \{\mathbf{x}\}$. See the definition at the beginning of this section.

13. False. The weights in the linear combination must sum to one. See the definition.

14. True. Theorem 2.

15. True. See equation (1).

16. True. See the definition prior to Theorem 3.

17. False. A flat is a translate of a subspace. See the definition prior to Theorem 3.

18. False. A flat of dimension 2 is called a hyperplane only if the flat is considered a subset of $\mathbb{R}^3$. In general, a hyperplane is a flat of dimension $n - 1$. See the definition prior to Theorem 3.

19. True. A hyperplane in $\mathbb{R}^3$ has dimension 2, so it is a plane. See the definition prior to Theorem 3.

20. True. A flat through the origin is a subspace translated by the $\mathbf{0}$ vector.

21. Span $\{\mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1\}$ is a plane if and only if $\{\mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1\}$ is linearly independent. Suppose c_2 and c_3 satisfy $c_2(\mathbf{v}_2 - \mathbf{v}_1) + c_3(\mathbf{v}_3 - \mathbf{v}_1) = \mathbf{0}$. Then $c_2\mathbf{v}_2 + c_3\mathbf{v}_3 - (c_2 + c_3)\mathbf{v}_1 = \mathbf{0}$. Then $c_2 = c_3 = 0$, because $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set. This shows that $\{\mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1\}$ is a linearly independent set. Thus, Span $\{\mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1\}$ is a plane in $\mathbb{R}^3$.

22. Since $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$, the set W = Span $\{\mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1\}$ is a plane in $\mathbb{R}^3$, by Exercise 21. Thus, $W + \mathbf{v}_1$ is a plane parallel to W that contains $\mathbf{v}_1$. Since $\mathbf{v}_2 = (\mathbf{v}_2 - \mathbf{v}_1) + \mathbf{v}_1$, $W + \mathbf{v}_1$ contains $\mathbf{v}_2$. Similarly, $W + \mathbf{v}_1$ contains $\mathbf{v}_3$. Finally, Theorem 1 shows that aff $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is the plane $W + \mathbf{v}_1$ that contains $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

23. Let $S = \{\mathbf{x} : A\mathbf{x} = \mathbf{b}\}$. To show that S is affine, it suffices to show that S is a flat, by Theorem 3. Let $W = \{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}$. Then W is a subspace of $\mathbb{R}^n$, by Theorem 2 in Section 4.2 (or Theorem 12 in Section 2.8). Since $S = W + \mathbf{p}$, where $\mathbf{p}$ satisfies $A\mathbf{p} = \mathbf{b}$, by Theorem 6 in Section 1.5, S is a translate of W, and hence S is a flat.

24. Suppose $\mathbf{p}, \mathbf{q} \in S$ and $t \in \mathbb{R}$. Then, by properties of the dot product (Theorem 1 in Section 6.1),

$$[(1 - t)\mathbf{p} + t\,\mathbf{q}] \cdot \mathbf{v} = (1 - t)(\mathbf{p} \cdot \mathbf{v}) + t\,(\mathbf{q} \cdot \mathbf{v}) = (1 - t)k + t\,k = k$$

Thus, $[(1 - t)\mathbf{p} + t\,\mathbf{q}] \in S$, by definition of S. This shows that S is an affine set.

25. A suitable set consists of any three vectors that are not collinear and have 5 as their third entry. If 5 is their third entry, they lie in the plane $x_3 = 5$. If the vectors are not collinear, their affine hull cannot be a line, so it must be the plane. For example use $S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} \right\}$.

26. A suitable set consists of any four vectors that lie in the plane $2x_1 + x_2 - 3x_3 = 12$ and are not collinear. If the vectors are not collinear, their affine hull cannot be a line, so it must be the plane. For example use $S = \left\{ \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 12 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix} \right\}$.

27. If $\mathbf{p}, \mathbf{q} \in f(S)$, then there exist $\mathbf{r}, \mathbf{s} \in S$ such that $f(\mathbf{r}) = \mathbf{p}$ and $f(\mathbf{s}) = \mathbf{q}$. Given any $t \in \mathbb{R}$, we must show that $\mathbf{z} = (1 - t)\mathbf{p} + t\,\mathbf{q}$ is in $f(S)$. Since f is linear,
$\mathbf{z} = (1 - t)\mathbf{p} + t\,\mathbf{q} = (1 - t)f(\mathbf{r}) + t f(\mathbf{s}) = f((1 - t)\mathbf{r} + t\,\mathbf{s})$

Since S is affine, $(1 - t)\mathbf{r} + t\,\mathbf{s} \in S$. Thus, $\mathbf{z}$ is in $f(S)$ and $f(S)$ is affine.

28. Given an affine set T, let $S = \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \in T\}$. Consider $\mathbf{x}, \mathbf{y} \in S$ and $t \in \mathbb{R}$. Then

$$f((1 - t)\mathbf{x} + t\,\mathbf{y}) = (1 - t)f(\mathbf{x}) + t f(\mathbf{y})$$

But $f(\mathbf{x}) \in T$ and $f(\mathbf{y}) \in T$, so $(1 - t)f(\mathbf{x}) + t f(\mathbf{y}) \in T$ because T is an affine set. It follows that $[(1 - t)\mathbf{x} + t\,\mathbf{y}] \in S$. This is true for all $\mathbf{x}, \mathbf{y} \in S$ and $t \in \mathbb{R}$, so S is an affine set.

29. Since B is affine, Theorem 1 implies that B contains all affine combinations of points of B. Hence B contains all affine combinations of points of A. That is, aff $A \subseteq B$.

30. Since $B \subseteq$ aff B, we have $A \subseteq B \subseteq$ aff B. But aff B is an affine set, so Exercise 29 implies aff $A \subseteq$ aff B.

31. Since $A \subseteq (A \cup B)$, it follows from Exercise 30 that aff $A \subseteq$ aff $(A \cup B)$.

Similarly, aff $B \subseteq$ aff $(A \cup B)$, so [aff $A \cup$ aff B] $\subseteq$ aff $(A \cup B)$.

32. One possibility is to let $A = \{(0, 1), (0, 2)\}$ and $B = \{(1, 0), (2, 0)\}$. Then (aff A) $\cup$ (aff B) consists of the two coordinate axes, but aff $(A \cup B) = \mathbb{R}^2$.

33. Since $(A \cap B) \subseteq A$, it follows from Exercise 30 that aff $(A \cap B) \subseteq$ aff A.

Similarly, aff $(A \cap B) \subseteq$ aff B, so aff $(A \cap B) \subseteq$ (aff $A \cap$ aff B).

34. One possibility is to let $A = \{(0, 0), (0, 1)\}$ and $B = \{(0, 2), (0, 3)\}$. Then both aff A and aff B are equal to the y-axis. But $A \cap B = \varnothing$, so aff $(A \cap B) = \varnothing$.

8.2 - Affine Independence

Notes: Affine dependence and independence are developed in this section. Theorem 5 links affine independence to linear independence. This material has important applications to computer graphics.

1. .Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ -3 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 6 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$. Then $\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} -3 \\ 9 \end{bmatrix}, \mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Since $\mathbf{v}_3 - \mathbf{v}_1$ is a multiple of $\mathbf{v}_2 - \mathbf{v}_1$, these two points are linearly dependent. By Theorem 5, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is affinely dependent. Note that $(\mathbf{v}_2 - \mathbf{v}_1) - 3(\mathbf{v}_3 - \mathbf{v}_1) = \mathbf{0}$. A rearrangement produces the affine dependence relation $2\mathbf{v}_1 + \mathbf{v}_2 - 3\mathbf{v}_3 = \mathbf{0}$. (Note that the weights sum to zero.) Geometrically, $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are not collinear.

2. $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$. $\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix} -5 \\ -3 \end{bmatrix}$. Since $\mathbf{v}_3 - \mathbf{v}_1$ and $\mathbf{v}_2 - \mathbf{v}_1$ are not multiples, they are linearly independent. By Theorem 5, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is affinely independent.

3. The set is affinely independent. If the points are called $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$, then row reduction of $[\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \mathbf{v}_3 \ \ \mathbf{v}_4]$ shows that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$ and $\mathbf{v}_4 = 16\mathbf{v}_1 + 5\mathbf{v}_2 - 3\mathbf{v}_3$. Since there is unique way to write $\mathbf{v}_4$ in terms of the basis vectors, and the weights in the linear combination do not sum to one, $\mathbf{v}_4$ is not an affine combination of the first three vectors.

4. Name the points $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$. Then $\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 2 \\ -8 \\ 4 \end{bmatrix}$, $\mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix} 3 \\ -7 \\ -9 \end{bmatrix}$, $\mathbf{v}_4 - \mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ -6 \end{bmatrix}$. To study the linear independence of these points, row reduce the augmented matrix for $A\mathbf{x} = \mathbf{0}$:
$$\begin{bmatrix} 2 & 3 & 0 & 0 \\ -8 & -7 & 2 & 0 \\ 4 & -9 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & 5 & 2 & 0 \\ 0 & -15 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & 0 & 0 \\ 0 & 5 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -.6 & 0 \\ 0 & 1 & .4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$
The first three columns are linearly dependent, so $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is affinely dependent, by Theorem 5. To find the affine dependence relation, write the general solution of this system: $x_1 = .6x_3$, $x_2 = -.4x_3$, with x_3 free. Set $x_3 = 5$, for instance. Then $x_1 = 3$, $x_2 = -2$, and $x_3 = 5$. Thus, $3(\mathbf{v}_2 - \mathbf{v}_1) - 2(\mathbf{v}_3 - \mathbf{v}_1) + 5(\mathbf{v}_4 - \mathbf{v}_1) = \mathbf{0}$. Rearrange to obtain $-6\mathbf{v}_1 + 3\mathbf{v}_2 - 2\mathbf{v}_3 + 5\mathbf{v}_4 = \mathbf{0}$.

Alternative solution: Name the points $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$. Use Theorem 5(d) and study the homogeneous forms of the points. The first step is to move the bottom row of ones (in the augmented matrix) to the top to simplify the arithmetic:
$$[\tilde{\mathbf{v}}_1 \ \ \tilde{\mathbf{v}}_2 \ \ \tilde{\mathbf{v}}_3 \ \ \tilde{\mathbf{v}}_4] \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & 0 & 1 & -2 \\ 5 & -3 & -2 & 7 \\ 3 & 7 & -6 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1.2 \\ 0 & 1 & 0 & -.6 \\ 0 & 0 & 1 & .4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
Thus, $x_1 + 1.2x_4 = 0$, $x_2 - .6x_4 = 0$, and $x_3 + .4x_4 = 0$, with x_4 free. Take $x_4 = 5$, for example, and get $x_1 = -6$, $x_2 = 3$, and $x_3 = -2$. An affine dependence relation is $-6\mathbf{v}_1 + 3\mathbf{v}_2 - 2\mathbf{v}_3 + 5\mathbf{v}_4 = \mathbf{0}$.

5. $-4\mathbf{v}_1 + 5\mathbf{v}_2 - 4\mathbf{v}_3 + 3\mathbf{v}_4 = \mathbf{0}$ is an affine dependence relation. It can be found by row reducing the matrix $[\tilde{\mathbf{v}}_1 \ \ \tilde{\mathbf{v}}_2 \ \ \tilde{\mathbf{v}}_3 \ \ \tilde{\mathbf{v}}_4]$, and proceeding as in the solution to Exercise 4.

6. The set is affinely independent, as the following calculation with homogeneous forms shows:

$$\begin{bmatrix}\tilde{\mathbf{v}}_1 & \tilde{\mathbf{v}}_2 & \tilde{\mathbf{v}}_3 & \tilde{\mathbf{v}}_4\end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 3 \\ 3 & -1 & 5 & 5 \\ 1 & -2 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Alternative solution: Row reduction of $[\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \mathbf{v}_3 \;\; \mathbf{v}_4]$ shows that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$ and $\mathbf{v}_4 = -2\mathbf{v}_1 + 1.5\mathbf{v}_2 + 2.5\mathbf{v}_3$, but the weights in the linear combination do not sum to one, so this $\mathbf{v}_4$ is not an affine combination of the basis vectors and hence the set is affinely independent.

Note: A potential exam question might be to change the last entry of $\mathbf{v}_4$ from 0 to 1 and again ask if the set is affinely independent. Notice that row reduction of this new set of vectors $[\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \mathbf{v}_3 \;\; \mathbf{v}_4]$ shows that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$ and $\mathbf{v}_4 = -3\mathbf{v}_1 + \mathbf{v}_2 + 3\mathbf{v}_3$ is an affine combination of the basis.

7. Denote the given points as $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{p}$. Row reduce the augmented matrix for the equation $x_1\tilde{\mathbf{v}}_1 + x_2\tilde{\mathbf{v}}_2 + x_3\tilde{\mathbf{v}}_3 = \tilde{\mathbf{p}}$. Remember to move the bottom row of ones to the top as the first step to simplify the arithmetic by hand.

$$\begin{bmatrix}\tilde{\mathbf{v}}_1 & \tilde{\mathbf{v}}_2 & \tilde{\mathbf{v}}_3 & \tilde{\mathbf{p}}\end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 5 \\ -1 & 1 & 2 & 4 \\ 2 & 0 & -2 & -2 \\ 1 & 1 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus, $x_1 = -2$, $x_2 = 4$, $x_3 = -1$, and $\tilde{\mathbf{p}} = -2\tilde{\mathbf{v}}_1 + 4\tilde{\mathbf{v}}_2 - \tilde{\mathbf{v}}_3$, so $\mathbf{p} = -2\mathbf{v}_1 + 4\mathbf{v}_2 - \mathbf{v}_3$, and the barycentric coordinates are $(-2, 4, -1)$.

Alternative solution: Another way that this problem can be solved is by "translating" it to the origin. That is, compute $\mathbf{v}_2 - \mathbf{v}_1$, $\mathbf{v}_3 - \mathbf{v}_1$, and $\mathbf{p} - \mathbf{v}_1$, find weights c_2 and c_3 such that

$$c_2(\mathbf{v}_2 - \mathbf{v}_1) + c_3(\mathbf{v}_3 - \mathbf{v}_1) = \mathbf{p} - \mathbf{v}_1$$

and then write $\mathbf{p} = (1 - c_2 - c_3)\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$. Here are the calculations for Exercise 7:

$$\mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ -2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -4 \\ -1 \end{bmatrix}, \quad \mathbf{p} - \mathbf{v}_1 = \begin{bmatrix} 5 \\ 4 \\ -2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ -4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix}\mathbf{v}_2 - \mathbf{v}_1 & \mathbf{v}_3 - \mathbf{v}_1 & \mathbf{p} - \mathbf{v}_1\end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 \\ 2 & 3 & 5 \\ -2 & -4 & -4 \\ 0 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $\mathbf{p} - \mathbf{v}_1 = 4(\mathbf{v}_2 - \mathbf{v}_1) - 1(\mathbf{v}_3 - \mathbf{v}_1)$, and $\mathbf{p} = -2\,\mathbf{v}_1 + 4\mathbf{v}_2 - \mathbf{v}_3$.

8. Denote the given points as $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{p}$. Row reduce the augmented matrix for the equation $x_1\tilde{\mathbf{v}}_1 + x_2\tilde{\mathbf{v}}_2 + x_3\tilde{\mathbf{v}}_3 = \tilde{\mathbf{p}}$.

$$\begin{bmatrix}\tilde{\mathbf{v}}_1 & \tilde{\mathbf{v}}_2 & \tilde{\mathbf{v}}_3 & \tilde{\mathbf{p}}\end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 1 & 4 & 1 \\ -2 & 0 & -6 & -4 \\ 1 & 2 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus. $\tilde{\mathbf{p}} = 2\tilde{\mathbf{v}}_1 - \tilde{\mathbf{v}}_2 + 0\tilde{\mathbf{v}}_3$, so $\mathbf{p} = 2\mathbf{v}_1 - \mathbf{v}_2$. The barycentric coordinates are $(2, -1, 0)$.

Notice $\mathbf{v}_3 = 3\mathbf{v}_1 + \mathbf{v}_2$.

9. True. Theorem 5 uses the point $\mathbf{v}_1$ for the translation, but the paragraph after the theorem points out that any one of the points in the set can be used for the translation.

10. False. By Theorem 5, the set of homogeneous forms must be linearly dependent, too.

11. False, by (a) and (d) of Theorem 5.

12. True. If one statement in Theorem 5 is false, the other statements are false, too.

13. False. The weights in the linear combination must sum to zero, not one. See the definition at the beginning of this section.

14. False. Theorem 6 applies only when S is affinely independent.

15. False. The only points that have barycentric coordinates determined by S belong to aff S. See the definition after Theorem 6.

16. False. The color interpolation applies only to points whose barycentric coordinates are nonnegative, since the colors are formed by nonnegative combinations of red, green, and blue. See Example 5.

17. True. The barycentric coordinates have some zeros on the edges of the triangle and are only positive for interior points. See Example 6.

18. True. See the discussion of Fig. 5.

19. When a set of five points is translated by subtracting, say, the first point, the new set of four points must be linearly dependent, by Theorem 8 in Section 1.7, because the four points are in $\mathbb{R}^3$. By Theorem 5, the original set of five points is affinely dependent.

20. Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in $\mathbb{R}^n$ and $p \geq n + 2$. Since $p - 1 \geq n + 1$, the points $\mathbf{v}_2 - \mathbf{v}_1, \mathbf{v}_3 - \mathbf{v}_1, \ldots, \mathbf{v}_p - \mathbf{v}_1$ are linearly dependent, by Theorem 8 in Section 1.7. By Theorem 5, $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ is affinely dependent.

21. If $\{\mathbf{v}_1, \mathbf{v}_2\}$ is affinely dependent, then there exist c_1 and c_2, not both zero, such that $c_1 + c_2 = 0$, and $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = 0$. Then $c_1 = -c_2 \neq 0$ and $c_1\mathbf{v}_1 = -c_2\mathbf{v}_2 = c_1\mathbf{v}_2$, which implies that $\mathbf{v}_1 = \mathbf{v}_2$. Conversely, if $\mathbf{v}_1 = \mathbf{v}_2$, let $c_1 = 1$ and $c_2 = -1$. Then $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{v}_1 + (-1)\mathbf{v}_1 = \mathbf{0}$ and $c_1 + c_2 = 0$, which shows that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is affinely dependent.

22. Let S_1 consist of three (distinct) points on a line through the origin. The set is affinely dependent because the third point is on the line determined by the first two points. Let S_2 consist of two (distinct) points on a line through the origin. By Exercise 21, the set is affinely independent because the two points are distinct. (A correct solution should include a justification for the sets presented.)

23. a. The vectors $\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ are not multiples and hence are linearly independent. By Theorem 5, S is affinely independent.

b. $\mathbf{p}_1 \leftrightarrow \left(-\frac{6}{8}, \frac{9}{8}, \frac{5}{8}\right)$, $\mathbf{p}_2 \leftrightarrow \left(0, \frac{1}{2}, \frac{1}{2}\right)$, $\mathbf{p}_3 \leftrightarrow \left(\frac{14}{8}, -\frac{5}{8}, -\frac{1}{8}\right)$, $\mathbf{p}_4 \leftrightarrow \left(\frac{6}{8}, -\frac{5}{8}, \frac{7}{8}\right)$, $\mathbf{p}_5 \leftrightarrow \left(\frac{1}{4}, \frac{1}{8}, \frac{5}{8}\right)$

c. $\mathbf{p}_6$ is $(-, -, +)$, $\mathbf{p}_7$ is $(0, +, -)$, and $\mathbf{p}_8$ is $(+, +, -)$.

24. a. The vectors $\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ are not multiples and hence are linearly independent. By Theorem 5, S is affinely independent.

b. $\mathbf{p}_1 \leftrightarrow (-\frac{2}{7}, \frac{5}{7}, \frac{4}{7})$, $\mathbf{p}_2 \leftrightarrow (\frac{2}{7}, -\frac{5}{7}, \frac{10}{7})$, $\mathbf{p}_3 \leftrightarrow (\frac{2}{7}, \frac{2}{7}, \frac{3}{7})$

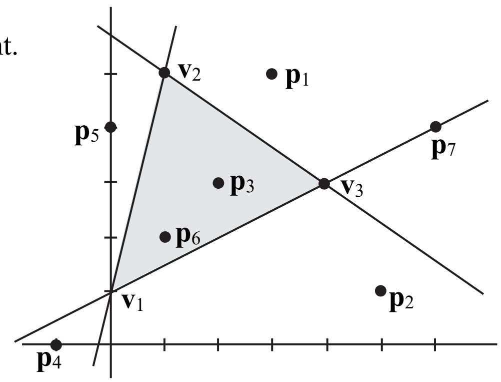
c. $\mathbf{p}_4 \leftrightarrow (+,-,-)$, $\mathbf{p}_5 \leftrightarrow (+,+,-)$,

$\mathbf{p}_6 \leftrightarrow (+,+,+)$, $\mathbf{p}_7 \leftrightarrow (-,0,+)$.

See the figure to the right. Actually,

$\mathbf{p}_4 \leftrightarrow (\frac{19}{14}, -\frac{2}{14}, -\frac{3}{14})$, $\mathbf{p}_5 \leftrightarrow (\frac{5}{14}, \frac{12}{14}, -\frac{3}{14})$,

$\mathbf{p}_6 \leftrightarrow (\frac{9}{14}, \frac{2}{14}, \frac{3}{14})$, $\mathbf{p}_7 \leftrightarrow (-\frac{1}{2}, 0, \frac{3}{2})$.



25. Suppose $S = \{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ is an affinely independent set. Then (7) has a solution, because $\mathbf{p}$ is in aff S. Hence (8) has a solution. By Theorem 5, the homogeneous forms of the points in S are linearly independent. Thus (8) has a unique solution. Then (7) also has a unique solution, because (8) encodes both equations that appear in (7).

The following argument mimics the proof of Theorem 7 in Section 4.4. If $S = \{\mathbf{b}_1, \dots, \mathbf{b}_k\}$ is an affinely independent set, then scalars $c_1, \dots, c_k$ exist that satisfy (7), by definition of aff S. Suppose $\mathbf{x}$ also has the representation

$$\mathbf{x} = d_1\mathbf{b}_1 + \cdots + d_k\mathbf{b}_k \quad \text{and} \quad d_1 + \cdots + d_k = 1 \qquad \text{(7a)}$$

for scalars $d_1, \dots, d_k$. Then subtraction produces the equation

$$\mathbf{0} = \mathbf{x} - \mathbf{x} = (c_1 - d_1)\mathbf{b}_1 + \cdots + (c_k - d_k)\mathbf{b}_k \qquad \text{(7b)}$$

The weights in (7b) sum to zero because the c's and the d's separately sum to one. This is impossible, unless each weight in (8) is zero, because S is an affinely independent set. This proves that $c_i = d_i$ for $i = 1, \dots, k$.

26. Let $\mathbf{p} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Then $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{x}{a}\begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} + \frac{y}{b}\begin{bmatrix} 0 \\ b \\ 0 \end{bmatrix} + \frac{z}{c}\begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix} + \left(1 - \frac{x}{a} - \frac{y}{b} - \frac{z}{c}\right)\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ So the barycentric coordinates are x/a, y/b, z/c, and $1 - x/a - y/b - z/c$. This holds for any nonzero choices of a, b, and c.

27. If $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is an affinely dependent set, then there exist scalars c_1, c_2, and c_3, not all zero, such that $c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{0}$ and $c_1 + c_2 + c_3 = 0$. But then, applying the transformation f,

$$c_1 f(\mathbf{p}_1) + c_2 f(\mathbf{p}_2) + c_3 f(\mathbf{p}_3) = f(c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3) = f(\mathbf{0}) = \mathbf{0},$$

since f is linear. This shows that $\{f(\mathbf{p}_1), f(\mathbf{p}_2), f(\mathbf{p}_3)\}$ is also affinely dependent.

28. If the translated set $\{\mathbf{p}_1 + \mathbf{q}, \mathbf{p}_2 + \mathbf{q}, \mathbf{p}_3 + \mathbf{q}\}$ were affinely dependent, then there would exist real numbers c_1, c_2, and c_3, not all zero and with $c_1 + c_2 + c_3 = 0$, such that

$c_1(\mathbf{p}_1 + \mathbf{q}) + c_2(\mathbf{p}_2 + \mathbf{q}) + c_3(\mathbf{p}_3 + \mathbf{q}) = \mathbf{0}$.

But then,

$c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 + (c_1 + c_2 + c_3)\mathbf{q} = \mathbf{0}$.

Since $c_1 + c_2 + c_3 = 0$, this implies $c_1\mathbf{p}_1 + c_2\mathbf{p}_2 + c_3\mathbf{p}_3 = \mathbf{0}$, which would make $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ affinely dependent. But $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ is affinely independent, so the translated set must in fact be affinely independent, too.

29. Let $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, and $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. Then $\det[\tilde{\mathbf{a}}\ \ \tilde{\mathbf{b}}\ \ \tilde{\mathbf{c}}] = \det\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 1 & 1 & 1 \end{bmatrix} = \det\begin{bmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{bmatrix}$,

by using the transpose property of the determinant (Theorem 5 in Section 3.2). By Exercise 30 in Section 3.3, this determinant equals 2 times the area of the triangle with vertices at **a**, **b**, and **c**.

30. If **p** is on the line through **a** and **b**, then **p** is an affine combination of **a** and **b**, so $\tilde{\mathbf{p}}$ is a linear combination of $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$. Thus the columns of $[\tilde{\mathbf{a}}\ \ \tilde{\mathbf{b}}\ \ \tilde{\mathbf{p}}]$ are linearly dependent. So the determinant of this matrix is zero.

31. If $[\tilde{\mathbf{a}}\ \ \tilde{\mathbf{b}}\ \ \tilde{\mathbf{c}}]\begin{bmatrix} r \\ s \\ t \end{bmatrix} = \tilde{\mathbf{p}}$, then Cramer's rule gives $r = \det[\tilde{\mathbf{p}}\ \ \tilde{\mathbf{b}}\ \ \tilde{\mathbf{c}}]\,/\det[\tilde{\mathbf{a}}\ \ \tilde{\mathbf{b}}\ \ \tilde{\mathbf{c}}]$. By Exercise 29, the numerator of this quotient is twice the area of $\Delta\mathbf{pbc}$, and the denominator is twice the area of $\Delta\mathbf{abc}$. This proves the formula for r. The other formulas are proved using Cramer's rule for s and t.

32. Let $\mathbf{p} = (1 - x)\mathbf{q} + x\mathbf{a}$, where **q** is on the line segment from **b** to **c**. Then, because the determinant is a linear function of the first column when the other columns are fixed (Section 3.2),

$\det[\tilde{\mathbf{p}}\ \ \tilde{\mathbf{b}}\ \ \tilde{\mathbf{c}}] = \det[(1 - x)\tilde{\mathbf{q}} + x\tilde{\mathbf{a}}\ \ \tilde{\mathbf{b}}\ \ \tilde{\mathbf{c}}] = (1 - x)\cdot\det[\tilde{\mathbf{q}}\ \ \tilde{\mathbf{b}}\ \ \tilde{\mathbf{c}}] + x\cdot\det[\tilde{\mathbf{a}}\ \ \tilde{\mathbf{b}}\ \ \tilde{\mathbf{c}}]$.

Now, $[\tilde{\mathbf{q}}\ \ \tilde{\mathbf{b}}\ \ \tilde{\mathbf{c}}]$ is a singular matrix because $\tilde{\mathbf{q}}$ is a linear combination of $\tilde{\mathbf{b}}$ and $\tilde{\mathbf{c}}$. So

$\det[\tilde{\mathbf{q}}\ \ \tilde{\mathbf{b}}\ \ \tilde{\mathbf{c}}] = 0$ and $\det[\tilde{\mathbf{p}}\ \ \tilde{\mathbf{b}}\ \ \tilde{\mathbf{c}}] = x\cdot\det[\tilde{\mathbf{a}}\ \ \tilde{\mathbf{b}}\ \ \tilde{\mathbf{c}}]$.

33. $\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 6 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}$, and $\mathbf{a} - \mathbf{v}_1 = \begin{bmatrix} -1 \\ -3 \\ 15 \end{bmatrix}$. Solve $\begin{bmatrix} \mathbf{v}_2 - \mathbf{v}_1 & \mathbf{v}_3 - \mathbf{v}_1 & -\mathbf{b} \end{bmatrix}\begin{bmatrix} c_2 \\ c_3 \\ t \end{bmatrix} = \mathbf{a} - \mathbf{v}_1$.

$$\begin{bmatrix} 6 & 2 & -1.4 & -1 \\ 0 & 6 & -1.5 & -3 \\ 1 & 4 & 3.1 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 3.1 & 15 \\ 0 & 6 & -1.5 & -3 \\ 0 & -22 & -20 & -91 \end{bmatrix} \sim \begin{bmatrix} 1 & 4 & 3.1 & 15 \\ 0 & 2 & -0.5 & -1 \\ 0 & -22 & -20 & -91 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 4.1 & 17 \\ 0 & 2 & -0.5 & -1 \\ 0 & 0 & -25.5 & -102 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0.6 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$ Thus, $c_2 = 0.6$, $c_3 = 0.5$, and $t = 4$.

The intersection point is $\mathbf{x}(4) = \mathbf{a} + 4\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 9 \end{bmatrix} + 4\begin{bmatrix} 1.4 \\ 1.5 \\ -3.1 \end{bmatrix} = \begin{bmatrix} 5.6 \\ 6.0 \\ -3.4 \end{bmatrix}$ and

$$\mathbf{x}(4) = (1-0.6-0.5)\mathbf{v}_1 + 0.6\mathbf{v}_2 + 0.5\mathbf{v}_3 = -0.1\begin{bmatrix}1\\3\\-6\end{bmatrix} + 0.6\begin{bmatrix}7\\3\\-5\end{bmatrix} + 0.5\begin{bmatrix}3\\9\\-2\end{bmatrix} = \begin{bmatrix}5.6\\6.0\\-3.4\end{bmatrix}.$$

The first barycentric coordinate is negative, so the intersection point is not inside the triangle.

34. $\mathbf{v}_2-\mathbf{v}_1 = \begin{bmatrix}7\\0\\-1\end{bmatrix}$, $\mathbf{v}_3-\mathbf{v}_1 = \begin{bmatrix}2\\8\\2\end{bmatrix}$, and $\mathbf{a}-\mathbf{v}_1 = \begin{bmatrix}-1\\-2\\12\end{bmatrix}$. Solve $\begin{bmatrix}\mathbf{v}_2-\mathbf{v}_1 & \mathbf{v}_3-\mathbf{v}_1 & -\mathbf{b}\end{bmatrix}\begin{bmatrix}c_2\\c_3\\t\end{bmatrix} = \mathbf{a}-\mathbf{v}_1$.

$$\begin{bmatrix}7 & 2 & -0.9 & -1\\0 & 8 & -2 & -2\\-1 & 2 & 3.7 & 12\end{bmatrix} \sim \begin{bmatrix}1 & -2 & -3.7 & -12\\0 & 8 & -2 & -2\\0 & 16 & 25 & 83\end{bmatrix} \sim \begin{bmatrix}1 & -2 & -3.7 & -12\\0 & 4 & -1 & -1\\0 & 0 & 29 & 87\end{bmatrix}$$

$$\sim \begin{bmatrix}1 & -2 & -3.7 & -12\\0 & 4 & -1 & -1\\0 & 0 & 1 & 3\end{bmatrix} \sim \begin{bmatrix}1 & -2 & 0 & -0.9\\0 & 4 & 0 & 2\\0 & 0 & 1 & 3\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 0 & 0.1\\0 & 1 & 0 & 0.5\\0 & 0 & 1 & 3\end{bmatrix}$$

Thus, $c_2 = 0.1$, $c_3 = 0.5$, and $t = 3$.

The intersection point is $\mathbf{x}(3) = \mathbf{a} + 3\mathbf{b} = \begin{bmatrix}0\\0\\8\end{bmatrix} + 3\begin{bmatrix}0.9\\2.0\\-3.7\end{bmatrix} = \begin{bmatrix}2.7\\6.0\\-3.1\end{bmatrix}$ and

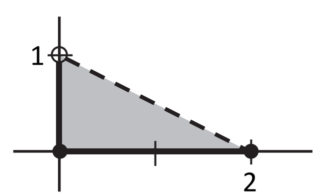
$$\mathbf{x}(3) = (1-0.1-0.5)\mathbf{v}_1 + 0.1\mathbf{v}_2 + 0.5\mathbf{v}_3 = 0.4\begin{bmatrix}1\\2\\-4\end{bmatrix} + 0.1\begin{bmatrix}8\\2\\-5\end{bmatrix} + 0.5\begin{bmatrix}3\\10\\-2\end{bmatrix} = \begin{bmatrix}2.7\\6.0\\-3.1\end{bmatrix}.$$

The barycentric coordinates are all positive, so the intersection point is inside the triangle.

8.3 - Convex Combinations

Notes: The notion of convexity is introduced in this section and has important applications in computer graphics. Bézier curves are introduced in Exercises 25-28 and explored in greater detail in Section 8.6.

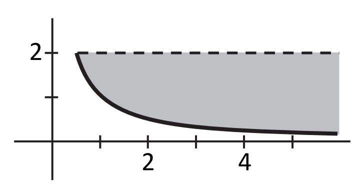
1. The set $V = \left\{\begin{bmatrix}0\\y\end{bmatrix} : 0 \le y < 1\right\}$ is the vertical line segment from (0,0) to (0,1) that includes (0,0) but not (0,1). The convex hull of S includes each line segment from a point in V to the point (2,0), as shown in the figure. The dashed line segment along the top of the shaded region indicates that this segment is not in conv S, because (0,1) is not in S.



2. a. Conv S includes all points **p** of the form

$$\mathbf{p} = (1-t)\begin{bmatrix}1/2\\2\end{bmatrix} + t\begin{bmatrix}x\\1/x\end{bmatrix} = \begin{bmatrix}1/2+t(x-1/2)\\2-t(2-1/x)\end{bmatrix}, \text{ where}$$

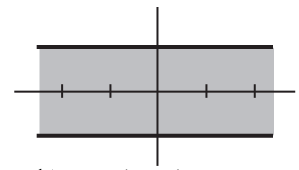
$x \ge 1/2$ and $0 \le t \le 1$. Notice that if $t = a/x$, then



and $\lim_{x\to\infty} \mathbf{p}(x) = \begin{bmatrix} 1/2+a \\ 2 \end{bmatrix}$, establishing that there are points arbitrarily close to the line $y = 2$ in conv S. Since the curve $y = 1/x$ is in S, the line segments between $y = 2$ and $y = 1/x$ are also included in conv S, whenever $x \ge 1/2$.

b. Recall that for any integer n, $\sin(x+2n\pi) = \sin(x)$. Then

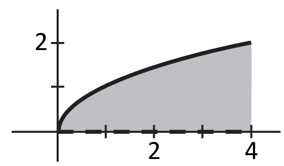
$$\mathbf{p} = (1-t)\begin{bmatrix} x \\ \sin(x) \end{bmatrix} + t\begin{bmatrix} x+2n\pi \\ \sin(x+2n\pi) \end{bmatrix} = \begin{bmatrix} x+2n\pi t \\ \sin(x) \end{bmatrix} \in \text{conv S.}$$



Notice that $\sin(x)$ is always a number between -1 and 1. For a fixed x and any real number r, an integer n and a number t (with $0 \le t \le 1$) can be chosen so that $r = x + 2n\pi t$.

c. Conv S includes all points **p** of the form

$$\mathbf{p} = (1-t)\begin{bmatrix} 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} x \\ \sqrt{x} \end{bmatrix} = t\begin{bmatrix} x \\ \sqrt{x} \end{bmatrix}, \text{ where } x \ge 0 \text{ and } 0 \le t \le 1.$$



Letting t = a/x, $\lim_{x\to\infty} \mathbf{p} = \begin{bmatrix} a \\ 0 \end{bmatrix}$ establishing that there are points arbitrarily close to $y = 0$ in the set.

3. From Exercise 5, Section 8.1,

a. $\mathbf{p}_1 = 3\mathbf{b}_1 - \mathbf{b}_2 - \mathbf{b}_3 \notin$ conv S since some of the coefficients are negative.

b. $\mathbf{p}_2 = 2\mathbf{b}_1 + 0\mathbf{b}_2 + \mathbf{b}_3 \notin$ conv S since the coefficients do not sum to one.

c. $\mathbf{p}_3 = -\mathbf{b}_1 + 2\mathbf{b}_2 + 0\mathbf{b}_3 \notin$ conv S since some of the coefficients are negative.

4. From Exercise 5, Section 8.1,

a. $\mathbf{p}_1 = -4\mathbf{b}_1 + 2\mathbf{b}_2 + 3\mathbf{b}_3 \notin$ conv S since some of the coefficients are negative.

b. $\mathbf{p}_2 = .2\mathbf{b}_1 + .5\mathbf{b}_2 + .3\mathbf{b}_3 \in$ conv S since the coefficients are nonnegative and sum to one.

c. $\mathbf{p}_3 = \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3 \notin$ conv S since the coefficients do not sum to one.

5. Row reduce the matrix $[\tilde{\mathbf{v}}_1 \quad \tilde{\mathbf{v}}_2 \quad \tilde{\mathbf{v}}_3 \quad \tilde{\mathbf{v}}_4 \quad \tilde{\mathbf{p}}_1 \quad \tilde{\mathbf{p}}_2]$ to obtain the barycentric coordinates

$\mathbf{p}_1 = -\frac{1}{6}\mathbf{v}_1 + \frac{1}{3}\mathbf{v}_2 + \frac{2}{3}\mathbf{v}_3 + \frac{1}{6}\mathbf{v}_4$, so $\mathbf{p}_1 \notin$ conv S, and $\mathbf{p}_2 = \frac{1}{3}\mathbf{v}_1 + \frac{1}{3}\mathbf{v}_2 + \frac{1}{6}\mathbf{v}_3 + \frac{1}{6}\mathbf{v}_4$, so $\mathbf{p}_2 \in$ conv S.

6. Let W be the subspace spanned by the orthogonal set $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. As in Example 1, the barycentric coordinates of the points $\mathbf{p}_1, \ldots, \mathbf{p}_4$ with respect to S are easy to compute, and they determine whether or not a point is in Span S, aff S, or conv S.

a. $\text{proj}_W \, \mathbf{p}_1 = \dfrac{\mathbf{p}_1 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \dfrac{\mathbf{p}_1 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 + \dfrac{\mathbf{p}_1 \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3}\mathbf{v}_3$

$$= \frac{1}{2}\begin{bmatrix} 2 \\ 0 \\ -1 \\ 2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 \\ -2 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -\frac{3}{2} \\ \frac{5}{2} \end{bmatrix} = \mathbf{p}_1$$

This shows that $\mathbf{p}_1$ is in W = Span S. Also, since the coefficients sum to 1, $\mathbf{p}_1$ is an aff S. However, $\mathbf{p}_1$ is not in conv S, because the coefficients are not all nonnegative.

b. Similarly, $\text{proj}_W\ \mathbf{p}_2 = \frac{\frac{9}{4}}{9}\mathbf{v}_1 + \frac{\frac{9}{4}}{9}\mathbf{v}_2 + \frac{\frac{9}{2}}{9}\mathbf{v}_3 = \frac{1}{4}\mathbf{v}_1 + \frac{1}{4}\mathbf{v}_2 + \frac{1}{2}\mathbf{v}_3 = \mathbf{p}_2$. This shows that $\mathbf{p}_2$ lies in Span *S*. Also, since the coefficients sum to 1, $\mathbf{p}_2$ is in aff *S*. In fact, $\mathbf{p}_2$ is in conv *S*, because the coefficients are also nonnegative.

c. $\text{proj}_W\ \mathbf{p}_3 = \frac{9}{9}\mathbf{v}_1 + \frac{9}{9}\mathbf{v}_2 - \frac{18}{9}\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2 - 2\mathbf{v}_3 = \mathbf{p}_3$. Thus $\mathbf{p}_3$ is in Span *S*. However, since the coefficients do not sum to one, $\mathbf{p}_3$ is not in aff *S* and certainly not in conv *S*.

d. $\text{proj}_W\ \mathbf{p}_4 = \frac{6}{9}\mathbf{v}_1 + \frac{8}{9}\mathbf{v}_2 + \frac{8}{9}\mathbf{v}_3 \neq \mathbf{p}_4$. Since $\text{proj}_W\ \mathbf{p}_4$ is the closest point in Span *S* to $\mathbf{p}_4$, the point $\mathbf{p}_4$ is not in Span *S*. In particular, $\mathbf{p}_4$ cannot be in aff *S* or conv *S*.

7. $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $\mathbf{p}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{p}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\mathbf{p}_3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\mathbf{p}_4 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, $T = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$

a. Use an augmented matrix (with four augmented columns) to write the homogeneous forms of $\mathbf{p}_1, \ldots, \mathbf{p}_4$ in terms of the homogeneous forms of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, with the first step interchanging rows 1 and 3:

$$\begin{bmatrix} \tilde{\mathbf{v}}_1 & \tilde{\mathbf{v}}_2 & \tilde{\mathbf{v}}_3 & \tilde{\mathbf{p}}_1 & \tilde{\mathbf{p}}_2 & \tilde{\mathbf{p}}_3 & \tilde{\mathbf{p}}_4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & 2 & 4 & 2 & 3 & 2 & 0 \\ 0 & 3 & 1 & 1 & 2 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{6} & \frac{1}{2} & -\frac{1}{4} & \frac{3}{4} \\ 0 & 0 & 1 & \frac{1}{2} & \frac{1}{2} & \frac{3}{4} & -\frac{1}{4} \end{bmatrix}$$

The first four columns reveal that $\frac{1}{3}\tilde{\mathbf{v}}_1 + \frac{1}{6}\tilde{\mathbf{v}}_2 + \frac{1}{2}\tilde{\mathbf{v}}_3 = \tilde{\mathbf{p}}_1$ and $\frac{1}{3}\mathbf{v}_1 + \frac{1}{6}\mathbf{v}_2 + \frac{1}{2}\mathbf{v}_3 = \mathbf{p}_1$. Thus column 4 contains the barycentric coordinates of $\mathbf{p}_1$ relative to the triangle determined by *T*. Similarly, column 5 (as an augmented column) contains the barycentric coordinates of $\mathbf{p}_2$, column 6 contains the barycentric coordinates of $\mathbf{p}_3$, and column 7 contains the barycentric coordinates of $\mathbf{p}_4$.

b. $\mathbf{p}_3$ and $\mathbf{p}_4$ are outside conv *T*, because in each case at least one of the barycentric coordinates is negative. $\mathbf{p}_1$ is inside conv *T*, because all of its barycentric coordinates are positive. $\mathbf{p}_2$ is on the edge $\overline{\mathbf{v}_2\mathbf{v}_3}$ of conv *T*, because its barycentric coordinates are nonnegative and its first coordinate is 0.

8. a. The barycentric coordinates of $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$, and $\mathbf{p}_4$ are, respectively, $\left(\frac{12}{13}, \frac{3}{13}, -\frac{2}{13}\right)$, $\left(\frac{8}{13}, \frac{2}{13}, \frac{3}{13}\right)$, $\left(\frac{2}{3}, 0, \frac{1}{3}\right)$, and $\left(\frac{9}{13}, -\frac{1}{13}, \frac{5}{13}\right)$.

b. The point $\mathbf{p}_1$ and $\mathbf{p}_4$ are outside conv *T* since they each have a negative coordinate. The point $\mathbf{p}_2$ is inside conv *T* since the coordinates are positive, and $\mathbf{p}_3$ is on the edge $\overline{\mathbf{v}_1\mathbf{v}_3}$ of conv *T*.

9. The points $\mathbf{p}_1$ and $\mathbf{p}_3$ are outside the tetrahedron conv *S* since their barycentric coordinates contain negative numbers. The point $\mathbf{p}_2$ is on the face containing the vertices $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ since its first barycentric coordinate is zero and the rest are positive. The point $\mathbf{p}_4$ is inside conv *S* since all its barycentric coordinates are positive. The point $\mathbf{p}_5$ is on the edge between $\mathbf{v}_1$ and $\mathbf{v}_3$ since the first and third barycentric coordinates are positive and the rest are zero.

10. The point $\mathbf{q}_1$ is inside conv *S* because the barycentric coordinates are all positive. The point $\mathbf{q}_2$ is outside conv *S* because it has one negative barycentric coordinate. The point $\mathbf{q}_4$ is outside conv *S* for the same reason. The point $\mathbf{q}_3$ is on the edge between $\mathbf{v}_2$ and $\mathbf{v}_3$ because $\left(0, \frac{3}{4}, \frac{1}{4}, 0\right)$ shows that $\mathbf{q}_3$ is a convex combination of $\mathbf{v}_2$ and $\mathbf{v}_3$. The point $\mathbf{q}_5$ is on the face containing the vertices $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ because $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\right)$ shows that $\mathbf{q}_5$ is a convex combination of those vertices.

11. False. In order for $\mathbf{y}$ to be a convex combination, the c's must also all be nonnegative. See the definition at the beginning of this section.

12. True. See the definition prior to Theorem 7.

13. False. If S is convex, then conv S is equal to S. See Theorem 7.

14. True. Theorem 9.

15. False. The points do not have to be distinct. For example, S might consist of two points in $\mathbb{R}^5$. A point in conv S would be a convex combination of these two points. Caratheodory's Theorem requires $n + 1$ **or fewer** points.

16. False. For example, the union of two distinct points is not convex, but the individual points are.

17. If $\mathbf{p}, \mathbf{q} \in f(S)$, then there exist $\mathbf{r}, \mathbf{s} \in S$ such that $f(\mathbf{r}) = \mathbf{p}$ and $f(\mathbf{s}) = \mathbf{q}$. The goal is to show that the line segment $\mathbf{y} = (1 - t)\mathbf{p} + t\mathbf{q}$, for $0 \le t \le 1$, is in $f(S)$. Since f is linear,

$$\mathbf{y} = (1-t)\mathbf{p} + t\mathbf{q} = (1-t)f(\mathbf{r}) + tf(\mathbf{s}) = f((1-t)\mathbf{r} + t\mathbf{s})$$

Since S is convex, $(1 - t)\mathbf{r} + t\mathbf{s} \in S$ for $0 \le t \le 1$. Thus $\mathbf{y} \in f(S)$ and $f(S)$ is convex.

18. Suppose $\mathbf{r}, \mathbf{s} \in S$ and $0 \le t \le 1$. Then, since f is a linear transformation,

$$f[(1-t)\mathbf{r} + t\mathbf{s}] = (1-t)f(\mathbf{r}) + tf(\mathbf{s})$$

But $f(\mathbf{r}) \in T$ and $f(\mathbf{s}) \in T$, so $(1 - t)f(\mathbf{r}) + tf(\mathbf{s}) \in T$ since T is a convex set. It follows that $(1 - t)\mathbf{r} + t\mathbf{s} \in S$, because S consists of all points that f maps into T. This shows that S is convex.

19. It is straightforward to confirm the equations in the problem: (1) $\frac{1}{3}\mathbf{v}_1 + \frac{1}{3}\mathbf{v}_2 + \frac{1}{6}\mathbf{v}_3 + \frac{1}{6}\mathbf{v}_4 = \mathbf{p}$ and (2) $\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 - \mathbf{v}_4 = \mathbf{0}$. Notice that the coefficients of $\mathbf{v}_1$ and $\mathbf{v}_3$ in equation (2) are positive. With the notation of the proof of Caratheodory's Theorem, $d_1 = 1$ and $d_3 = 1$. The corresponding coefficients in equation (1) are $c_1 = \frac{1}{3}$ and $c_3 = \frac{1}{6}$. The ratios of these coefficients are $c_1 / d_1 = \frac{1}{3}$ and $c_3 / d_3 = \frac{1}{6}$. Use the smaller ratio to eliminate $\mathbf{v}_3$ from equation (1). That is, add $-\frac{1}{6}$ times equation (2) to equation (1):

$$\mathbf{p} = (\tfrac{1}{3} - \tfrac{1}{6})\mathbf{v}_1 + (\tfrac{1}{3} + \tfrac{1}{6})\mathbf{v}_2 + (\tfrac{1}{6} - \tfrac{1}{6})\mathbf{v}_3 + (\tfrac{1}{6} + \tfrac{1}{6})\mathbf{v}_4 = \tfrac{1}{6}\mathbf{v}_1 + \tfrac{1}{2}\mathbf{v}_2 + \tfrac{1}{3}\mathbf{v}_4$$

To obtain the second combination, multiply equation (2) by –1 to reverse the signs so that d_2 and d_4 become positive. Repeating the analysis with these terms eliminates the $\mathbf{v}_4$ term resulting in

$$\mathbf{p} = \tfrac{1}{2}\mathbf{v}_1 + \tfrac{1}{6}\mathbf{v}_2 + \tfrac{1}{3}\mathbf{v}_3.$$

20. $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ It is straightforward to confirm the equations in the problem: (1) $\frac{1}{121}\mathbf{v}_1 + \frac{72}{121}\mathbf{v}_2 + \frac{37}{121}\mathbf{v}_3 + \frac{1}{11}\mathbf{v}_4 = \mathbf{p}$ and (2) $10\mathbf{v}_1 - 6\mathbf{v}_2 + 7\mathbf{v}_3 - 11\mathbf{v}_4 = \mathbf{0}$.

Notice that the coefficients of $\mathbf{v}_1$ and $\mathbf{v}_3$ in equation (2) are positive. With the notation of the proof of Caratheodory's Theorem, $d_1 = 10$ and $d_3 = 7$. The corresponding coefficients in equation (1) are $c_1 = \frac{1}{121}$ and $c_3 = \frac{37}{121}$. The ratios of these coefficients are $c_1 / d_1 = \frac{1}{121} \div 10 = \frac{1}{1210}$ and $c_3 / d_3 = \frac{37}{121} \div 7 = \frac{37}{847}$. Use the smaller ratio to eliminate $\mathbf{v}_1$ from equation (1). That is, add $-\frac{1}{1210}$ times equation (2) to equation (1):

$$\mathbf{p} = (\tfrac{1}{121} - \tfrac{10}{1210})\mathbf{v}_1 + (\tfrac{72}{121} + \tfrac{6}{1210})\mathbf{v}_2 + (\tfrac{37}{121} - \tfrac{7}{1210})\mathbf{v}_3 + (\tfrac{1}{11} + \tfrac{11}{1210})\mathbf{v}_4 = \tfrac{3}{5}\mathbf{v}_2 + \tfrac{3}{10}\mathbf{v}_3 + \tfrac{1}{10}\mathbf{v}_4$$

To obtain the second combination, multiply equation (2) by –1 to reverse the signs so that d_2 and d_4 become positive. Repeating the analysis with these terms eliminates the $\mathbf{v}_4$ term resulting in

$$\mathbf{p} = (\tfrac{1}{121} + \tfrac{10}{121})\mathbf{v}_1 + (\tfrac{72}{121} - \tfrac{6}{121})\mathbf{v}_2 + (\tfrac{37}{121} + \tfrac{7}{121})\mathbf{v}_3 + (\tfrac{1}{11} - \tfrac{11}{121})\mathbf{v}_4 = \tfrac{1}{11}\mathbf{v}_1 + \tfrac{6}{11}\mathbf{v}_2 + \tfrac{4}{11}\mathbf{v}_3$$

21. Suppose $A \subseteq B$, where B is convex. Then, since B is convex, Theorem 7 implies that B contains all convex combinations of points of B. Hence B contains all convex combinations of points of A. That is, conv $A \subseteq B$.

22. Suppose $A \subseteq B$. Then $A \subseteq B \subseteq$ conv B. Since conv B is convex, Exercise 21 shows that conv $A \subseteq$ conv B.

23. a. Since $A \subseteq (A \cup B)$, Exercise 22 shows that conv $A \subseteq$ conv $(A \cup B)$. Similarly, conv $B \subseteq$ conv $(A \cup B)$. Thus, [(conv A) $\cup$ (conv B)] $\subseteq$ conv $(A \cup B)$.

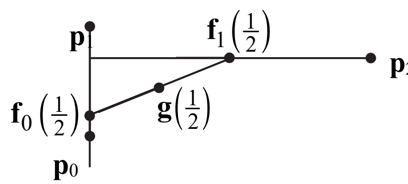
b. One possibility is to let A be two adjacent corners of a square and B be the other two corners. Then (conv A) $\cup$ (conv B) consists of two opposite sides of the square, but conv $(A \cup B)$ is the whole square.

24. a. Since $(A \cap B) \subseteq A$, Exercise 22 shows that conv $(A \cap B) \subseteq$ conv A. Similarly,

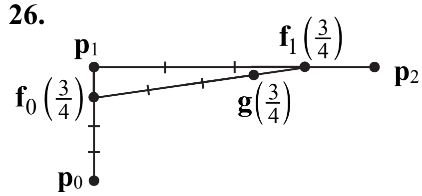
conv $(A \cap B) \subseteq$ conv B. Thus, conv $(A \cap B) \subseteq$ [(conv A) $\cap$ (conv B)].

b. One possibility is to let A be a pair of opposite vertices of a square and let B be the other pair of opposite vertices. Then conv A and conv B are intersecting diagonals of the square. $A \cap B$ is the empty set, so conv $(A \cap B)$ must be empty, too. But conv $A \cap$ conv B contains the single point where the diagonals intersect. So conv $(A \cap B)$ is a proper subset of cnv $A \cap$ conv B.

25.



26.



27. $\mathbf{g}(t) = (1-t)\mathbf{f}_0(t) + t\mathbf{f}_1(t)$
$= (1-t)[(1-t)\mathbf{p}_0 + t\mathbf{p}_1] + t[(1-t)\mathbf{p}_1 + t\mathbf{p}_2] = (1-t)^2\mathbf{p}_0 + 2t(1-t)\mathbf{p}_1 + t^2\mathbf{p}_2$.

The sum of the weights in the linear combination for $\mathbf{g}$ is $(1-t)^2 + 2t(1-t) + t^2$, which equals $(1-2t+t^2) + (2t-2t^2) + t^2 = 1$. The weights are each between 0 and 1 when $0 \le t \le 1$, so $\mathbf{g}(t)$ is in conv$\{\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2\}$.

28. $\mathbf{h}(t) = (1-t)\mathbf{g}_1(t) + t\mathbf{g}_2(t)$. Use the representation for $\mathbf{g}_1(t)$ from Exercise 27, and the analogous representation for $\mathbf{g}_2(t)$, based on the control points $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$, and obtain

$$\begin{aligned}
\mathbf{h}(t) &= (1-t)[(1-t)^2\mathbf{p}_0 + 2t(1-t)\mathbf{p}_1 + t^2\mathbf{p}_2] + t[(1-t)^2\mathbf{p}_1 + 2t(1-t)\mathbf{p}_2 + t^2\mathbf{p}_3] \\
&= (1-t)^3\mathbf{p}_0 + 2t(1-2t+t^2)\mathbf{p}_1 + (t^2-t^3)\mathbf{p}_2 + t(1-2t+t^2)\mathbf{p}_1 + 2t^2(1-t)\mathbf{p}_2 + t^3\mathbf{p}_3 \\
&= (1-3t+3t^2-t^3)\mathbf{p}_0 + (2t-4t^2+2t^3)\mathbf{p}_1 + (t^2-t^3)\mathbf{p}_2 \\
&\qquad + (t-2t^2+t^3)\mathbf{p}_1 + (2t^2-2t^3)\mathbf{p}_2 + t^3\mathbf{p}_3 \\
&= (1-3t+3t^2-t^3)\mathbf{p}_0 + (3t-6t^2+3t^3)\mathbf{p}_1 + (3t^2-3t^3)\mathbf{p}_2 + t^3\mathbf{p}_3
\end{aligned}$$

By inspection, the sum of the weights in this linear combination is 1, for all t. To show that the weights are nonnegative for $0 \le t \le 1$, factor the coefficients and write

$\mathbf{h}(t) = (1-t)^3\mathbf{p}_0 + 3t(1-t)^2\mathbf{p}_1 + 3t^2(1-t)\mathbf{p}_2 + t^3\mathbf{p}_3$ for $0 \le t \le 1$

Thus, $\mathbf{h}(t)$ is in the convex hull of the control points $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$.

8.4 - Hyperplanes

Notes: In this section lines and planes are generalized to higher dimensions using the notion of hyperplanes. Important topological ideas such as open, closed, and compact sets are introduced.

1. Let $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Then $\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}$. Choose $\mathbf{n}$ to be a vector orthogonal to $\mathbf{v}_2 - \mathbf{v}_1$, for example let $\mathbf{n} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Then $f(x_1, x_2) = 3x_1 + 4x_2$ and $d = f(\mathbf{v}_1) = 3(-1) + 4(4) = 13$.

This is easy to check by verifying that $f(\mathbf{v}_2)$ is also 13.

2. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$. Then $\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} -2 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \end{bmatrix}$. Choose $\mathbf{n}$ to be a vector orthogonal to $\mathbf{v}_2 - \mathbf{v}_1$, for example let $\mathbf{n} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$. Then $f(x_1, x_2) = 5x_1 - 3x_2$ and $d = f(\mathbf{v}_1) = 5(1) - 3(4) = -7$.

3. **a.** The set is open since it does not contain any of its boundary points.

b. The set is closed since it contains all of its boundary points.

c. The set is neither open nor closed since it contains some, but not all, of its boundary points.

d. The set is closed since it contains all of its boundary points.

e. The set is closed since it contains all of its boundary points.

4. **a**. The set is closed since it contains all of its boundary points.

b. The set is open since it does not contain any of its boundary points.

c. The set is neither open nor closed since it contains some, but not all, of its boundary points.

d. The set is closed since it contains all of its boundary points.

e. The set is open since it does not contain any of its boundary points.

5. **a.** The set is not compact since it is not closed, however it is convex.
b. The set is compact since it is closed and bounded. It is also convex.
c. The set is not compact since it is not closed, however it is convex.
d. The set is not compact since it is not bounded. It is not convex.
e. The set is not compact since it is not bounded, however it is convex.

6. **a.** The set is compact since it is closed and bounded. It is not convex.
b. The set is not compact since it is not closed. It is not convex.
c. The set is not compact since it is not closed, however it is convex.
d. The set is not compact since it is not bounded. It is convex.
e. The set is not compact since it is not closed. It is not convex.

7. a. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -1 \\ -2 \\ 5 \end{bmatrix}$, $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and compute the translated points

$$\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \quad \mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix}.$$

To solve the system of equations $(\mathbf{v}_2 - \mathbf{v}_1) \cdot \mathbf{n} = 0$ and $(\mathbf{v}_3 - \mathbf{v}_1) \cdot \mathbf{n} = 0$, reduce the augmented matrix for a system of two equations with three variables.

$$[1 \quad 3 \quad -2]\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0, \qquad [-2 \quad -3 \quad 2]\begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0.$$

Row operations show that $\begin{bmatrix} 1 & 3 & -2 & 0 \\ -2 & -3 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & -2 & 0 \end{bmatrix}$. A suitable normal vector is

$$\mathbf{n} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}.$$

b. The linear functional is $f(x_1, x_2, x_3) = 2x_2 + 3x_3$, so $d = f(1, 1, 3) = 2 + 9 = 11$. As a check, evaluate f at the other two points on the hyperplane: $f(2, 4, 1) = 8 + 3 = 11$ and $f(-1, -2, 5) = -4 + 15 = 11$.

8. **a**. Find a vector in the null space of the transpose of $[\mathbf{v}_2 - \mathbf{v}_1 \;\; \mathbf{v}_3 - \mathbf{v}_1]$. For example, take $\mathbf{n} = \begin{bmatrix} 4 \\ 3 \\ -6 \end{bmatrix}$.

b. $f(\mathbf{x}) = 4x_1 + 3x_2 - 6x_3$, $d = f(\mathbf{v}_1) = -8$

9. a. Find a vector in the null space of the transpose of $[\mathbf{v}_2 - \mathbf{v}_1 \;\; \mathbf{v}_3 - \mathbf{v}_1 \;\; \mathbf{v}_4 - \mathbf{v}_1]$. For example, take

$$\mathbf{n} = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix}.$$

b. $f(\mathbf{x}) = 3x_1 - x_2 + 2x_3 + x_4$, $d = f(\mathbf{v}_1) = 5$

10. **a**. Find a vector in the null space of the transpose of $[\mathbf{v}_2 - \mathbf{v}_1 \;\; \mathbf{v}_3 - \mathbf{v}_1 \;\; \mathbf{v}_4 - \mathbf{v}_1]$. For example, take

$$\mathbf{n} = \begin{bmatrix} -2 \\ 3 \\ -5 \\ 1 \end{bmatrix}.$$

b. $f(\mathbf{x}) = -2x_1 + 3x_2 - 5x_3 + x_4$, $d = f(\mathbf{v}_1) = 4$

11. $\mathbf{n} \cdot \mathbf{p} = 2$; $\mathbf{n} \cdot \mathbf{0} = 0 < 2$; $\mathbf{n} \cdot \mathbf{v}_1 = 5 > 2$; $\mathbf{n} \cdot \mathbf{v}_2 = -2 < 2$; $\mathbf{n} \cdot \mathbf{v}_3 = 2$. Hence $\mathbf{v}_2$ is on the same side of H as $\mathbf{0}$, $\mathbf{v}_1$ is on the other side, and $\mathbf{v}_3$ is in H.

12. Let $H = [f : d]$, where $f(x_1, x_2, x_3) = 3x_1 + x_2 - 2x_3$. $f(\mathbf{a}_1) = -5$, $f(\mathbf{a}_2) = 4$. $f(\mathbf{a}_3) = 3$, $f(\mathbf{b}_1) = 7$,

$f(\mathbf{b}_2) = 4$, and $f(\mathbf{b}_3) = 6$. Choose $d = 4$ so that all the points in A are in or on one side of H and all the points in B are in or on the other side of H. There is no hyperplane parallel to H that strictly separates A and B because both sets have a point at which f takes on the value of 4. There may be (and in fact is) a hyperplane that is *not* parallel to H that strictly separates A and B.

13. $H_1 = \{\mathbf{x} : \mathbf{n}_1 \cdot \mathbf{x} = d_1\}$ and $H_2 = \{\mathbf{x} : \mathbf{n}_2 \cdot \mathbf{x} = d_2\}$. Since $\mathbf{p}_1 \in H_1$, $d_1 = \mathbf{n}_1 \cdot \mathbf{p}_1 = 4$. Similarly, $d_2 = \mathbf{n}_2 \cdot \mathbf{p}_2 = 22$. Solve the simultaneous system $[1\ \ 2\ \ 4\ \ 2]\mathbf{x} = 4$ and $[2\ \ 3\ \ 1\ \ 5]\mathbf{x} = 22$:

$$\begin{bmatrix} 1 & 2 & 4 & 2 & 4 \\ 2 & 3 & 1 & 5 & 22 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -10 & 4 & 32 \\ 0 & 1 & 7 & -1 & -14 \end{bmatrix}$$

The general solution provides one set of vectors, $\mathbf{p}$, $\mathbf{v}_1$, and $\mathbf{v}_2$. Other choices are possible.

$$\mathbf{x} = \begin{bmatrix} 32 \\ -14 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 10 \\ -7 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \mathbf{p} + x_3\mathbf{v}_1 + x_4\mathbf{v}_2, \quad \text{where } \mathbf{p} = \begin{bmatrix} 32 \\ -14 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_1 = \begin{bmatrix} 10 \\ -7 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Then $H_1 \cap H_2 = \{\mathbf{x} : \mathbf{x} = \mathbf{p} + x_3\mathbf{v}_1 + x_4\mathbf{v}_2\}$.

14. Since each of F_1 and F_2 can be described as the solution sets of $A_1\mathbf{x} = \mathbf{b}_1$ and $A_2\mathbf{x} = \mathbf{b}_2$ respectively, where A_1 and A_2 have rank 2, their intersection is described as the solution set to $\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}\mathbf{x} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}$.

Since $2 \le \text{rank}\left(\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}\right) \le 4$, the solution set will have dimensions $6-2 = 4$, $6-3 = 3$, or $6-4 = 2$.

15. $f(x_1, x_2, x_3, x_4) = A\mathbf{x} = x_1 - 3x_2 + 4x_3 - 2x_4$ and $d = b = 5$

16. $f(x_1, x_2, x_3, x_4, x_5) = A\mathbf{x} = 2x_1 + 5x_2 - 3x_3 + 6x_5$ and $d = b = 0$

17. Since by Theorem 3 in Section 6.1, Row $B = (\text{Nul } B)^{\perp}$, choose a nonzero vector $\mathbf{n} \in \text{Nul } B$. For example take $\mathbf{n} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. Then $f(x_1, x_2, x_3) = x_1 - 2x_2 + x_3$ and $d = 0$

18. Since by Theorem 3, Section 6.1, Row $B = (\text{Nul } B)^{\perp}$, choose a nonzero vector $\mathbf{n} \in \text{Nul } B$. For example take $\mathbf{n} = \begin{bmatrix} -11 \\ 4 \\ 1 \end{bmatrix}$. Then $f(x_1, x_2, x_3) = -11x_1 + 4x_2 + x_3$ and $d = 0$

19. Theorem 3 in Section 6.1 says that $(\text{Col } B)^{\perp} = \text{Nul } B^T$. Since the two columns of B are clearly linear independent, the rank of B is 2, as is the rank of B^T. So dim Nul $B^T = 1$, by the Rank Theorem, since there are three columns in B^T. This means that Nul B^T is one-dimensional and any nonzero vector $\mathbf{n}$ in Nul B^T will be orthogonal to H and can be used as its normal vector. Solve the linear system $B^T\mathbf{x} = \mathbf{0}$ by row reduction to find a basis for Nul B^T:

$$\begin{bmatrix} 1 & 4 & -7 & 0 \\ 0 & 2 & -6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & -3 & 0 \end{bmatrix} \Rightarrow \mathbf{n} = \begin{bmatrix} -5 \\ 3 \\ 1 \end{bmatrix}$$

Now, let $f(x_1, x_2, x_3) = -5x_1 + 3x_2 + x_3$. Since the hyperplane H is a subspace, it goes through the origin and d must be 0.

The solution is easy to check by evaluating f at each of the columns of B.

20. Since by Theorem 3, Section 6.1, Col B = (Nul $B^T)^\perp$, choose a nonzero vector $\mathbf{n}$ in Nul B^T. For example take $\mathbf{n} = \begin{bmatrix} -6 \\ 2 \\ 1 \end{bmatrix}$. Then $f(x_1, x_2, x_3) = -6x_1 + 2x_2 + x_3$ and $d = 0$

21. False. A linear functional goes from $\mathbb{R}^n$ to $\mathbb{R}$. See the definition at the beginning of this section.

22. True. See the statement after (3).

23. False. See the discussion of (1) and (4). There is a $1\times n$ matrix A such that $f(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

Equivalently, there is a point $\mathbf{n}$ in $\mathbb{R}^n$ such that $f(\mathbf{x}) = \mathbf{n}\cdot\mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$.

24. False. The vector $\mathbf{n}$ must be nonzero. If $\mathbf{n} = \mathbf{0}$, then the given set is empty if $d \neq 0$ and the set is all of $\mathbb{R}^n$ if $d = 0$.

25. True. See the comments after the definition of *strictly separate*.

26. False. Theorem 12 requires that the sets A and B be convex. For example, A could be the boundary of a circle and B could be the center of the circle.

27. False. See the sets in Figure 4.

28. False. Some other hyperplane might strictly separate them. See the caution at the end of Example 8.

29. Notice that the side of the triangle closest to $\mathbf{p}$ is $\overline{\mathbf{v}_2\mathbf{v}_3}$. A vector orthogonal to $\overline{\mathbf{v}_2\mathbf{v}_3}$ is $\mathbf{n} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

Take $f(x_1, x_2) = 3x_1 - 2x_2$. Then $f(\mathbf{v}_2) = f(\mathbf{v}_3) = 9$ and $f(\mathbf{p}) = 10$ so any d satisfying $9 < d < 10$ will work. There are other possible answers.

30. Notice that the side of the triangle closest to $\mathbf{p}$ is $\overline{\mathbf{v}_1\mathbf{v}_3}$ A vector orthogonal to $\overline{\mathbf{v}_1\mathbf{v}_3}$ is $\mathbf{n} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

Take $f(x_1, x_2) = -2x_1 + 3x_2$. Then $f(\mathbf{v}_1) = f(\mathbf{v}_3) = 4$ and $f(\mathbf{p}) = 5$ so any d satisfying $4 < d < 5$ will work. There are other possible answers.

31. Let L be the line segment from the center of $B(\mathbf{0}, 3)$ to the center of $B(\mathbf{p}, 1)$. This is on the line through the origin in the direction of $\mathbf{p}$. The length of L is $(4^2 + 1^2)^{1/2} \approx 4.1231$. This exceeds the sum of the radii of the two disks, so the disks do not touch. If the disks did touch, there would be no hyperplane (line) strictly separating them, but the line orthogonal to L through the point of tangency would (weakly) separate them. Since the disks are separated slightly, the hyperplane need not be exactly perpendicular to L, but the easiest one to find is a hyperplane H whose normal vector is $\mathbf{p}$. So define f by $f(\mathbf{x}) = \mathbf{p}\cdot\mathbf{x}$.

To find d, evaluate f at any point on L that is between the two disks. If the disks were tangent, that point would be three-fourths of the distance between their centers, since the radii are 3 and 1. Since the disks are slightly separated, the distance is about 4.1231. Three-fourths of this distance is greater than 3, and one-fourth of this distance is greater than 1. A suitable value of d is $f(\mathbf{q})$, where $\mathbf{q} = (.25)\mathbf{0} + (.75)\mathbf{p} = (3, .75)$. So $d = \mathbf{p}\cdot\mathbf{q} = 4(3) + 1(.75) = 12.75$.

32. The normal to the separating hyperplane has the direction of the line segment between $\mathbf{p}$ and $\mathbf{q}$. So, let $\mathbf{n} = \mathbf{p} - \mathbf{q} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$. The distance between $\mathbf{p}$ and $\mathbf{q}$ is $\sqrt{20}$, which is more than the sum of the radii of the two balls. The large ball has center $\mathbf{q}$. A point three-fourths of the distance from $\mathbf{q}$ to $\mathbf{p}$ will be greater than 3 units from $\mathbf{q}$ and greater than 1 unit from $\mathbf{p}$. This point is

$$\mathbf{x} = .75\mathbf{p} + .25\mathbf{q} = .75\begin{bmatrix} 6 \\ 1 \end{bmatrix} + .25\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5.0 \\ 1.5 \end{bmatrix}$$

Compute $\mathbf{n}\cdot\mathbf{x} = 17$. The desired hyperplane is $\left\{\begin{bmatrix} x \\ y \end{bmatrix} : 4x - 2y = 17\right\}$.

33. Exercise 2(a) in Section 8.3 gives one possibility. Or let $S = \{(x, y) : x^2y^2 = 1 \text{ and } y > 0\}$. Then conv S is the upper (open) half-plane.

34. One possibility is $B = \{(x, y) : x^2y^2 = 1 \text{ and } y > 0\}$ and $A = \{(x, y) : |x| \le 1 \text{ and } y = 0\}$.

35. Let $\mathbf{x}, \mathbf{y} \in B(\mathbf{p}, \delta)$ and suppose $\mathbf{z} = (1 - t)\mathbf{x} + t\mathbf{y}$, where $0 \le t \le 1$. Then

$$\|\mathbf{z} - \mathbf{p}\| = \|[(1 - t)\mathbf{x} + t\mathbf{y}] - \mathbf{p}\| = \|(1 - t)(\mathbf{x} - \mathbf{p}) + t(\mathbf{y} - \mathbf{p})\|$$
$$\le (1 - t)\,\|\mathbf{x} - \mathbf{p}\| + t\|\mathbf{y} - \mathbf{p}\| < (1 - t)\,\delta + t\delta = \delta$$

where the first inequality comes from the Triangle Inequality (Theorem 17 in Section 6.7) and the second inequality follows from $\mathbf{x}, \mathbf{y} \in B(\mathbf{p}, \delta)$. It follows that $\mathbf{z} \in B(\mathbf{p}, \delta)$ and $B(\mathbf{p}, \delta)$ is convex.

36. Let S be a bounded set. Then there exists a $\delta > 0$ such that $S \subseteq B(\mathbf{0}, \delta)$. But $B(\mathbf{0}, \delta)$ is convex by Exercise 35, so Theorem 9 in Section 8.3 (or Exercise 21 in Section 8.3) implies that conv $S \subseteq B(\mathbf{p}, \delta)$ and conv S is bounded.

8.5 - Polytopes

Notes: A polytope is the convex hull of a finite number of points. Polytopes and simplices are important in linear programming, which has numerous applications in engineering design and business management. The behavior of functions on polytopes is studied in this section.

1. Evaluate each linear functional at each of the three extreme points of S. Then select the extreme point(s) that give the maximum value of the functional.
a. $f(\mathbf{p}_1) = 1$, $f(\mathbf{p}_2) = -1$, and $f(\mathbf{p}_3) = -3$, so $m = 1$ at $\mathbf{p}_1$.
b. $f(\mathbf{p}_1) = 1$, $f(\mathbf{p}_2) = 5$, and $f(\mathbf{p}_3) = 1$, so $m = 5$ at $\mathbf{p}_2$.
c. $f(\mathbf{p}_1) = -3$, $f(\mathbf{p}_2) = -3$, and $f(\mathbf{p}_3) = 5$, so $m = 5$ at $\mathbf{p}_3$.

2. Evaluate each linear functional at each of the three extreme points of S. Then select the point(s) that give the maximum value of the functional.
a. $f(\mathbf{p}_1) = -1$, $f(\mathbf{p}_2) = 3$, and $f(\mathbf{p}_3) = 3$, so $m = 3$ on the set conv$\{\mathbf{p}_2, \mathbf{p}_3\}$.

b. $f(\mathbf{p}_1) = 1$, $f(\mathbf{p}_2) = 1$, and $f(\mathbf{p}_3) = -1$, so $m = 1$ on the set conv$\{\mathbf{p}_1, \mathbf{p}_2\}$.
c. $f(\mathbf{p}_1) = -1$, $f(\mathbf{p}_2) = -3$, and $f(\mathbf{p}_3) = 0$, so $m = 0$ at $\mathbf{p}_3$.

3. Evaluate each linear functional at each of the three extreme points of S. Then select the point(s) that give the minimum value of the functional.
a. $f(\mathbf{p}_1) = 1$, $f(\mathbf{p}_2) = -1$, and $f(\mathbf{p}_3) = -3$, so $m = -3$ at the point $\mathbf{p}_3$
b. $f(\mathbf{p}_1) = 1$, $f(\mathbf{p}_2) = 5$, and $f(\mathbf{p}_3) = 1$, so $m = 1$ on the set conv$\{\mathbf{p}_1, \mathbf{p}_3\}$.
c. $f(\mathbf{p}_1) = -3$, $f(\mathbf{p}_2) = -3$, and $f(\mathbf{p}_3) = 5$, so $m = -3$ on the set conv $\{\mathbf{p}_1, \mathbf{p}_2\}$.

4. Evaluate each linear functional at each of the three extreme points of S. Then select the point(s) that give the minimum value of the functional.
a. $f(\mathbf{p}_1) = -1$, $f(\mathbf{p}_2) = 3$, and $f(\mathbf{p}_3) = 3$, so $m = -1$ at the point $\mathbf{p}_1$.
b. $f(\mathbf{p}_1) = 1$, $f(\mathbf{p}_2) = 1$, and $f(\mathbf{p}_3) = -1$, so $m = -1$ at the point $\mathbf{p}_3$.
c. $f(\mathbf{p}_1) = -1$, $f(\mathbf{p}_2) = -3$, and $f(\mathbf{p}_3) = 0$, so $m = -3$ at the point $\mathbf{p}_2$.

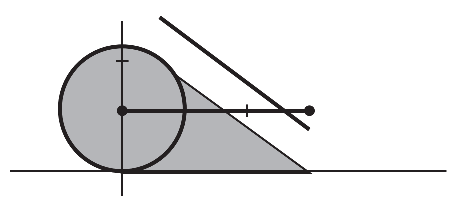
5. The two inequalities are (a) $x_1 + 2x_2 \le 10$ and (b) $3x_1 + x_2 \le 15$. Line (a) goes from (0,5) to (10,0). Line (b) goes from (0,15) to (5,0). One vertex is (0,0). The x_1-intercepts (when $x_2 = 0$) are 10 and 5, so (5,0) is a vertex. The x_2-intercepts (when $x_1 = 0$) are 5 and 15, so (0,5) is a vertex. The two lines intersect at (4,3) so (4,3) is a vertex. The minimal representation is $\left\{\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}5\\0\end{bmatrix}, \begin{bmatrix}4\\3\end{bmatrix}, \begin{bmatrix}0\\5\end{bmatrix}\right\}$

6. The two inequalities are (a) $2x_1 + 3x_2 \le 18$ and (b) $4x_1 + x_2 \le 16$. Line (a) goes from (0,6) to (9,0). Line (b) goes from (0,16) to (4,0). One vertex is (0,0). The x_1-intercepts (when $x_2 = 0$) are 9 and 4, so (4,0) is a vertex. The x_2-intercepts (when $x_1 = 0$) are 6 and 16, so (0,6) is a vertex. The two lines intersect at (3,4) so (3,4) is a vertex. The minimal representation is $\left\{\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}4\\0\end{bmatrix}, \begin{bmatrix}3\\4\end{bmatrix}, \begin{bmatrix}0\\6\end{bmatrix}\right\}$

7. The three inequalities are (a) $x_1 + 3x_2 \le 18$, (b) $x_1 + x_2 \le 10$, and (c) $4x_1 + x_2 \le 28$. Line (a) goes from (0,6) to (18,0). Line (b) goes from (0,10) to (10,0). And line (c) goes from (0,28) to (7,0). One vertex is (0,0). The x_1-intercepts (when $x_2 = 0$) are 18, 10, and 7, so (7,0) is a vertex. The x_2-intercepts (when $x_1 = 0$) are 6, 10, and 28, so (0,6) is a vertex. All three lines go through (6,4), so (6,4) is a vertex. The minimal representation is $\left\{\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}7\\0\end{bmatrix}, \begin{bmatrix}6\\4\end{bmatrix}, \begin{bmatrix}0\\6\end{bmatrix}\right\}$.

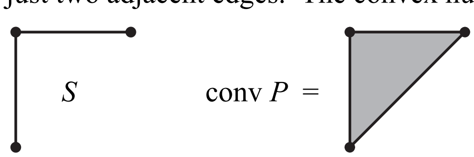
8. The three inequalities are (a) $2x_1 + x_2 \le 8$, (b) $x_1 + x_2 \le 6$, and (c) $x_1 + 2x_2 \le 7$. Line (a) goes from (0,8) to (4,0). Line (b) goes from (0,6) to (6,0). And line (c) goes from (0,3.5) to (7,0). One vertex is (0,0). The x_1-intercepts (when $x_2 = 0$) are 4, 6, and 7, so (4,0) is a vertex. The x_2-intercepts (when $x_1 = 0$) are 8, 6, and 3.5, so (0,3.5) is a vertex. All three lines go through (3,2), so (3,2) is a vertex. The minimal representation is $\left\{\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}4\\0\end{bmatrix}, \begin{bmatrix}3\\2\end{bmatrix}, \begin{bmatrix}0\\3.5\end{bmatrix}\right\}$

9. The origin is an extreme point, but it is not a vertex. It is an extreme point since it is not in the interior of any line segment that lies in S. It is not a vertex since the only supporting hyperplane (line) containing the origin also contains the line segment from (0,0) to (3,0).



10. One possibility is a ray. It has an extreme point at one end.

11. ne possibility is to let S be a square that includes part of the boundary but not all of it. For example, include just two adjacent edges. The convex hull of the profile P is a triangular region.



12. a. $f_0(S^5) = 6$, $f_1(S^5) = 15$, $f_2(S^5) = 20$, $f_3(S^5) = 15$, $f_4(S^5) = 6$, and $6 - 15 + 20 - 15 + 6 = 2$.

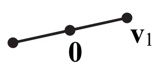
b.

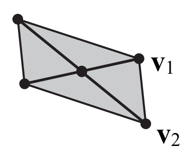
	f_0	f_1	f_2	f_3	f_4
S^1	2				
S^2	3	3			
S^3	4	6	4		
S^4	5	10	10	5	
S^5	6	15	20	15	6

$f_k(S^n) = \begin{pmatrix} n+1 \\ k+1 \end{pmatrix}$, where $\begin{pmatrix} a \\ b \end{pmatrix} = \dfrac{a!}{b!(a-b)!}$ is the binomial coefficient.

13. a. To determine the number of k-faces of the 5-dimensional hypercube C^5, look at the pattern that is followed in building C^4 from C^3. For example, the 2-faces in C^4 include the 2-faces of C^3 and the 2-faces in the translated image of C^3. In addition, there are the 1-faces of C^3 that are "stretched" into 2-faces. In general, the number of k-faces in C^n equals twice the number of k-faces in C^{n-1} plus the number of $(k-1)$-faces in C^{n-1}. Here is the pattern: $f_k(C^n) = 2f_k(C^{n-1}) + f_{k-1}(C^{n-1})$. For $k = 0, 1, \ldots, 4$, and $n = 5$, this gives $f_0(C^5) = 32$, $f_1(C^5) = 80$, $f_2(C^5) = 80$, $f_3(C^5) = 40$, and $f_4(C^5) = 10$. These numbers satisfy Euler's formula since, $32 - 80 + 80 - 40 + 10 = 2$.

b. The general formula is $f_k(C^n) = 2^{n-k}\begin{pmatrix} n \\ k \end{pmatrix}$, where $\begin{pmatrix} a \\ b \end{pmatrix} = \dfrac{a!}{b!(a-b)!}$ is the binomial coefficient.

14. a. X^1 is a line segment 

X^2 is a parallelogram 

b. $f_0(X^3) = 6$, $f_1(X^3) = 12$, $f_2(X^3) = 8$. X^3 is an octahedron.
c. $f_0(X^4) = 8$, $f_1(X^4) = 24$, $f_2(X^4) = 32$, $f_3(X^4) = 16$, $8 - 24 + 32 - 16 = 0$
d. $f_k(X^n) = 2^{k+1}\begin{pmatrix} n \\ k+1 \end{pmatrix}$, $0 \le k \le n-1$, where $\begin{pmatrix} a \\ b \end{pmatrix} = \dfrac{a!}{b!(a-b)!}$ is the binomial coefficient.

15. a. $f_0(P^n) = f_0(Q) + 1$
b. $f_k(P^n) = f_k(Q) + f_{k-1}(Q)$
c. $f_{n-1}(P^n) = f_{n-2}(Q) + 1$

16. True. See the definition at the beginning of this section.

17. False. It has six facets (faces).

18. True. See the definition after Example 1.

19. True. See Theorem 14.

20. False. S must be compact. See Theorem 15.

21. False. The maximum is always attained at some extreme point, but there may be other points that are not extreme points at which the maximum is also attained. See Theorem 16.

22. True. See the comment before Fig. 7.

23. True. Follows from Euler's formula with $n = 2$.

24. Let $\mathbf{v}$ be an extreme point of the convex set S and let $T = \{\mathbf{y} \in S : \mathbf{y} \neq \mathbf{v}\}$. If $\mathbf{y}$ and $\mathbf{z}$ are in T, then $\overline{\mathbf{yz}} \subseteq S$ since S is convex. But since $\mathbf{v}$ is an extreme point of S, $\mathbf{v} \notin \overline{\mathbf{yz}}$, so $\overline{\mathbf{yz}} \subset T$. Thus T is convex.

Conversely, suppose $\mathbf{v} \in S$, but $\mathbf{v}$ is not an extreme point of S. Then there exist $\mathbf{y}$ and $\mathbf{z}$ in S such that $\mathbf{v} \in \overline{\mathbf{yz}}$, with $\mathbf{v} \neq \mathbf{y}$ and $\mathbf{v} \neq \mathbf{z}$. It follows that $\mathbf{y}$ and $\mathbf{z}$ are in T, but $\overline{\mathbf{yz}} \not\subset T$. Hence T is not convex.

25. Let S be convex and let $\mathbf{x} \in cS + dS$, where $c > 0$ and $d > 0$. Then there exist $\mathbf{s}_1$ and $\mathbf{s}_2$ in S such that $\mathbf{x} = c\mathbf{s}_1 + d\mathbf{s}_2$. But then $\mathbf{x} = c\mathbf{s}_1 + d\mathbf{s}_2 = (c+d)\left(\dfrac{c}{c+d}\mathbf{s}_1 + \dfrac{d}{c+d}\mathbf{s}_2\right)$. Since S is convex, $\dfrac{c}{c+d}\mathbf{s}_1 + \dfrac{d}{c+d}\mathbf{s}_2 \in S$ and $\mathbf{x} \in (c + d)S$. For the converse, pick an typical point in $(c + d)S$ and show it is in $cS + dS$.

26. For example, let $S = \{1, 2\}$ in $\mathbb{R}^1$. Then $2S = \{2, 4\}$, $3S = \{3, 6\}$ and $(2 + 3)S = \{5, 10\}$. However, $2S + 3S = \{2, 4\} + \{3, 6\} = \{2 + 3, 4 + 3, 2 + 6, 4 + 6\} = \{5, 7, 8, 10\} \neq (2 + 3)S$.

27. Suppose A and B are convex. Let $\mathbf{x}, \mathbf{y} \in A + B$. Then there exist $\mathbf{a}, \mathbf{c} \in A$ and $\mathbf{b}, \mathbf{d} \in B$ such that $\mathbf{x} = \mathbf{a} + \mathbf{b}$ and $\mathbf{y} = \mathbf{c} + \mathbf{d}$. For any t such that $0 \leq t \leq 1$, we have

$$\begin{aligned}\mathbf{w} = (1-t)\mathbf{x} + t\mathbf{y} &= (1-t)(\mathbf{a}+\mathbf{b}) + t(\mathbf{c}+\mathbf{d}) \\ &= \left[(1-t)\mathbf{a} + t\mathbf{c}\right] + \left[(1-t)\mathbf{b} + t\mathbf{d}\right]\end{aligned}$$

But $(1 - t)\mathbf{a} + t\mathbf{c} \in A$ since A is convex, and $(1 - t)\mathbf{b} + t\mathbf{d} \in B$ since B is convex. Thus $\mathbf{w}$ is in $A + B$, which shows that $A + B$ is convex.

28. **a.** Since each edge belongs to two facets, kr is twice the number of edges: $kr = 2e$. Since each edge has two vertices, $sv = 2e$.

b. $v - e + r = 2$, so $\frac{2e}{s} - e + \frac{2e}{k} = 2 \;\Rightarrow\; \frac{1}{s} + \frac{1}{k} = \frac{1}{2} + \frac{1}{e}$

c. A polygon must have at least three sides, so $k \geq 3$. At least three edges meet at each vertex, so $s \geq 3$. But both k and s cannot both be greater than 3, for then the left side of the equation in (b) could not exceed $1/2$.

When $k = 3$, we get $\frac{1}{s} - \frac{1}{6} = \frac{1}{e}$, so $s = 3, 4,$ or 5. For these values, we get $e = 6, 12,$ or 30, corresponding to the tetrahedron, the octahedron, and the icosahedron, respectively.

When $s = 3$, we get $\frac{1}{k} - \frac{1}{6} = \frac{1}{e}$, so $k = 3, 4,$ or 5 and $e = 6, 12,$ or 30, respectively. These values correspond to the tetrahedron, the cube, and the dodecahedron.

8.6 - Curves and Surfaces

Notes: This section moves beyond lines and planes to the study of some of the curves that are used to model surfaces in engineering and computer aided design. Notice that these curves have a matrix representation.

1. The original curve is $\mathbf{x}(t) = (1 - t)^3\mathbf{p}_0 + 3t(1 - t)^2\mathbf{p}_1 + 3t^2(1 - t)\mathbf{p}_2 + t^3\mathbf{p}_3$ $(0 \leq t \leq 1)$. Since the curve is determined by its control points, it seems reasonable that to translate the curve, one should translate the control points. In this case, the new Bézier curve $\mathbf{y}(t)$ would have the equation

$$\mathbf{y}(t) = (1 - t)^3(\mathbf{p}_0 + \mathbf{b}) + 3t(1 - t)^2(\mathbf{p}_1 + \mathbf{b}) + 3t^2(1 - t)(\mathbf{p}_2 + \mathbf{b}) + t^3(\mathbf{p}_3 + \mathbf{b})$$

$$= (1 - t)^3\mathbf{p}_0 + 3t(1 - t)^2\mathbf{p}_1 + 3t^2(1 - t)\mathbf{p}_2 + t^3\mathbf{p}_3 + (1 - t)^3\mathbf{b} + 3t(1 - t)^2\mathbf{b} + 3t^2(1 - t)\mathbf{b} + t^3\mathbf{b}$$

A routine algebraic calculation verifies that $(1 - t)^3 + 3t(1 - t)^2 + 3t^2(1 - t) + t^3 = 1$ for all t. Thus $\mathbf{y}(t) = \mathbf{x}(t) + \mathbf{b}$ for all t, and translation by $\mathbf{b}$ maps a Bézier curve into a Bézier curve.

2. a. Equation (15) reveals that each polynomial weight is nonnegative for $0 \le t \le 1$, since $4 - 3t > 0$. For the sum of the coefficients, use (15) with the first term expanded: $1 - 3t + 6t^2 - t^3$. The 1 here plus the 4 and 1 in the coefficients of $\mathbf{p}_1$ and $\mathbf{p}_2$, respectively, sum to 6, while the other terms sum to 0. This explains the 1/6 in the formula for $\mathbf{x}(t)$, which makes the coefficients sum to 1. Thus, $\mathbf{x}(t)$ is a convex combination of the control points for $0 \le t \le 1$.

b. Since the coefficients inside the brackets in equation (14) sum to 6, it follows that

$\mathbf{b} = \frac{1}{6}\left[6\mathbf{b}\right] = \frac{1}{6}\left[(1-t)^3\mathbf{b} + (3t^3 - 6t^2 + 4)\mathbf{b} + (-3t^3 + 3t^2 + 3t + 1)\mathbf{b} + t^3\mathbf{b}\right]$ and hence $\mathbf{x}(t) + \mathbf{b}$ may be written in a similar form, with $\mathbf{p}_i$ replaced by $\mathbf{p}_i + \mathbf{b}$ for each i. This shows that $\mathbf{x}(t) + \mathbf{b}$ is a cubic B-spline with control points $\mathbf{p}_i + \mathbf{b}$ for $i = 0, \ldots, 3$.

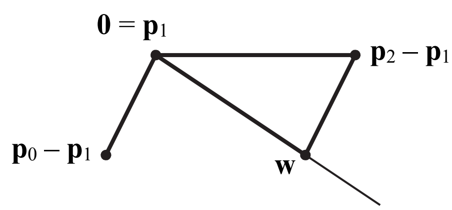
3. a. $\mathbf{x}'(t) = (-3 + 6t - 3t^2)\mathbf{p}_0 + (3 - 12t + 9t^2)\mathbf{p}_1 + (6t - 9t^2)\mathbf{p}_2 + 3t^2\mathbf{p}_3$, so $\mathbf{x}'(0) = -3\mathbf{p}_0 + 3\mathbf{p}_1 = 3(\mathbf{p}_1 - \mathbf{p}_0)$, and $\mathbf{x}'(1) = -3\mathbf{p}_2 + 3\mathbf{p}_3 = 3(\mathbf{p}_3 - \mathbf{p}_2)$. This shows that the tangent vector $\mathbf{x}'(0)$ points in the direction from $\mathbf{p}_0$ to $\mathbf{p}_1$ and is three times the length of $\mathbf{p}_1 - \mathbf{p}_0$. Likewise, $\mathbf{x}'(1)$ points in the direction from $\mathbf{p}_2$ to $\mathbf{p}_3$ and is three times the length of $\mathbf{p}_3 - \mathbf{p}_2$. In particular, $\mathbf{x}'(1) = \mathbf{0}$ if and only if $\mathbf{p}_3 = \mathbf{p}_2$.

b. $\mathbf{x}''(t) = (6 - 6t)\mathbf{p}_0 + (-12 + 18t)\mathbf{p}_1 + (6 - 18t)\mathbf{p}_2 + 6t\mathbf{p}_3$, so that

$$\mathbf{x}''(0) = 6\mathbf{p}_0 - 12\mathbf{p}_1 + 6\mathbf{p}_2 = 6(\mathbf{p}_0 - \mathbf{p}_1) + 6(\mathbf{p}_2 - \mathbf{p}_1)$$

and $\mathbf{x}''(1) = 6\mathbf{p}_1 - 12\mathbf{p}_2 + 6\mathbf{p}_3 = 6(\mathbf{p}_1 - \mathbf{p}_2) + 6(\mathbf{p}_3 - \mathbf{p}_2)$

For a picture of $\mathbf{x}''(0)$, construct a coordinate system with the origin at $\mathbf{p}_1$, temporarily, label $\mathbf{p}_0$ as $\mathbf{p}_0 - \mathbf{p}_1$, and label $\mathbf{p}_2$ as $\mathbf{p}_2 - \mathbf{p}_1$. Finally, construct a line from this new origin through the sum of $\mathbf{p}_0 - \mathbf{p}_1$ and $\mathbf{p}_2 - \mathbf{p}_1$, extended out a bit. That line points in the direction of $\mathbf{x}''(0)$.



$$\mathbf{w} = (\mathbf{p}_0 - \mathbf{p}_1) + (\mathbf{p}_2 - \mathbf{p}_1) = \tfrac{1}{6}\mathbf{x}''(0)$$

4. a. $\mathbf{x}'(t) = \frac{1}{6}\left[\left(-3t^2 + 6t - 3\right)\mathbf{p}_0 + \left(9t^2 - 12t\right)\mathbf{p}_1 + \left(-9t^2 + 6t + 3\right)\mathbf{p}_2 + 3t^2\mathbf{p}_3\right]$

$\mathbf{x}'(0) = \frac{1}{2}(\mathbf{p}_2 - \mathbf{p}_0)$ and $\mathbf{x}'(1) = \frac{1}{2}(\mathbf{p}_3 - \mathbf{p}_1)$

(Verify that, in the first part of Fig. 10, a line drawn through $\mathbf{p}_0$ and $\mathbf{p}_2$ is parallel to the tangent line at the beginning of the B-spline.)

When $\mathbf{x}'(0)$ and $\mathbf{x}'(1)$ are both zero, the figure collapses and the convex hull of the set of control points is the line segment between $\mathbf{p}_0$ and $\mathbf{p}_3$, in which case $\mathbf{x}(t)$ is a straight line. Where does $\mathbf{x}(t)$ start? In this case,

$$\mathbf{x}(t) = \tfrac{1}{6}\left[(-4t^3 + 6t^2 + 2)\mathbf{p}_0 + (4t^3 - 6t^2 + 4)\mathbf{p}_3\right]$$

$\mathbf{x}(0) = \frac{1}{3}\mathbf{p}_0 + \frac{2}{3}\mathbf{p}_3$ and $\mathbf{x}(1) = \frac{2}{3}\mathbf{p}_0 + \frac{1}{3}\mathbf{p}_3$

The curve begins closer to $\mathbf{p}_3$ and finishes closer to $\mathbf{p}_0$. Could it turn around during its travel? Since $\mathbf{x}'(t) = 2t(1 - t)(\mathbf{p}_0 - \mathbf{p}_3)$, the curve travels in the direction $\mathbf{p}_0 - \mathbf{p}_3$, so when $\mathbf{x}'(0) = \mathbf{x}'(1) = 0$, the curve always moves away from $\mathbf{p}_3$ toward $\mathbf{p}_0$ for $0 \le t \le 1$.

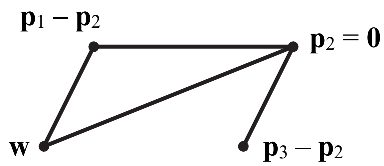
b. $\mathbf{x}''(t) = (1 - t)\mathbf{p}_0 + (-2 + 3t)\mathbf{p}_1 + (1 - 3t)\mathbf{p}_2 + t\mathbf{p}_3$

$\mathbf{x}''(0) = \mathbf{p}_0 - 2\mathbf{p}_1 + \mathbf{p}_2 = (\mathbf{p}_0 - \mathbf{p}_1) + (\mathbf{p}_2 - \mathbf{p}_1)$

and $\mathbf{x}''(1) = \mathbf{p}_1 - 2\mathbf{p}_2 + \mathbf{p}_3 = (\mathbf{p}_1 - \mathbf{p}_2) + (\mathbf{p}_3 - \mathbf{p}_2)$

For a picture of $\mathbf{x}''(0)$, construct a coordinate system with the origin at $\mathbf{p}_1$, temporarily, label $\mathbf{p}_0$ as $\mathbf{p}_0 - \mathbf{p}_1$, and label $\mathbf{p}_2$ as $\mathbf{p}_2 - \mathbf{p}_1$. Finally, construct a line from this new origin to the sum of $\mathbf{p}_0 - \mathbf{p}_1$ and $\mathbf{p}_2 - \mathbf{p}_1$. That segment represents $\mathbf{x}''(0)$.

For a picture of $\mathbf{x}''(1)$, construct a coordinate system with the origin at $\mathbf{p}_2$, temporarily, label $\mathbf{p}_1$ as $\mathbf{p}_1 - \mathbf{p}_2$, and label $\mathbf{p}_3$ as $\mathbf{p}_3 - \mathbf{p}_2$. Finally, construct a line from this new origin to the sum of $\mathbf{p}_1 - \mathbf{p}_2$ and $\mathbf{p}_3 - \mathbf{p}_2$. That segment represents $\mathbf{x}''(1)$.



$\mathbf{w} = (\mathbf{p}_1 - \mathbf{p}_2) + (\mathbf{p}_3 - \mathbf{p}_2) = \mathbf{x}''(1)$

5. **a**. From Exercise 3(a) or equation (9) in the text,
$\mathbf{x}'(1) = 3(\mathbf{p}_3 - \mathbf{p}_2)$
Use the formula for $\mathbf{x}'(0)$, with the control points from $\mathbf{y}(t)$, and obtain
$\mathbf{y}'(0) = -3\mathbf{p}_3 + 3\mathbf{p}_4 = 3(\mathbf{p}_4 - \mathbf{p}_3)$
For C^1 continuity, $3(\mathbf{p}_3 - \mathbf{p}_2) = 3(\mathbf{p}_4 - \mathbf{p}_3)$, so $\mathbf{p}_3 = (\mathbf{p}_4 + \mathbf{p}_2)/2$, and $\mathbf{p}_3$ is the midpoint of the line segment from $\mathbf{p}_2$ to $\mathbf{p}_4$.

b. If $\mathbf{x}'(1) = \mathbf{y}'(0) = \mathbf{0}$, then $\mathbf{p}_2 = \mathbf{p}_3$ and $\mathbf{p}_3 = \mathbf{p}_4$. Thus, the "line segment" from $\mathbf{p}_2$ to $\mathbf{p}_4$ is just the point $\mathbf{p}_3$. [*Note*: In this case, the combined curve is still C^1 continuous, by definition. However, some choices of the other control points, $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{p}_5$, and $\mathbf{p}_6$ can produce a curve with a visible "corner" at $\mathbf{p}_3$, in which case the curve is not G^1 continuous at $\mathbf{p}_3$.]

6. **a**. With $\mathbf{x}(t)$ as in Exercise 2,
$\mathbf{x}(0) = (\mathbf{p}_0 + 4\mathbf{p}_1 + \mathbf{p}_2)/6$ and $\mathbf{x}(1) = (\mathbf{p}_1 + 4\mathbf{p}_2 + \mathbf{p}_3)/6$
Use the formula for $\mathbf{x}(0)$, but with the shifted control points for $\mathbf{y}(t)$, and obtain
$\mathbf{y}(0) = (\mathbf{p}_1 + 4\mathbf{p}_2 + \mathbf{p}_3)/6$
This equals $\mathbf{x}(1)$, so the B-spline is G^0 continuous at the join point.

b. From Exercise 4(a),

$\mathbf{x}'(1) = (\mathbf{p}_3 - \mathbf{p}_1)/2$ and $\mathbf{x}'(0) = (\mathbf{p}_2 - \mathbf{p}_0)/2$

Use the formula for $\mathbf{x}'(0)$ with the control points for $\mathbf{y}(t)$, and obtain

$\mathbf{y}'(0) = (\mathbf{p}_3 - \mathbf{p}_1)/2 = \mathbf{x}'(1)$
Thus the B-spline is C^1 continuous at the join point.

7. From Exercise 3(b),
$\mathbf{x}''(0) = 6(\mathbf{p}_0 - \mathbf{p}_1) + 6(\mathbf{p}_2 - \mathbf{p}_1)$ and $\mathbf{x}''(1) = 6(\mathbf{p}_1 - \mathbf{p}_2) + 6(\mathbf{p}_3 - \mathbf{p}_2)$

Use $\mathbf{x}''(0)$ with the control points for $\mathbf{y}(t)$, to get

$\mathbf{y}''(0) = 6(\mathbf{p}_3 - \mathbf{p}_4) + 6(\mathbf{p}_5 - \mathbf{p}_4)$

Set $\mathbf{x}''(1) = \mathbf{y}''(0)$ and divide by 6, to get

$$(\mathbf{p}_1 - \mathbf{p}_2) + (\mathbf{p}_3 - \mathbf{p}_2) = (\mathbf{p}_3 - \mathbf{p}_4) + (\mathbf{p}_5 - \mathbf{p}_4) \qquad (*)$$

Since the curve is C^1 continuous at $\mathbf{p}_3$, the point $\mathbf{p}_3$ is the midpoint of the segment from $\mathbf{p}_2$ to $\mathbf{p}_4$, by Exercise 5(a). Thus $\mathbf{p}_3 = \frac{1}{2}(\mathbf{p}_2 + \mathbf{p}_4)$, which leads to $\mathbf{p}_4 - \mathbf{p}_3 = \mathbf{p}_3 - \mathbf{p}_2$. Substituting into (*) gives

$$(\mathbf{p}_1 - \mathbf{p}_2) + (\mathbf{p}_3 - \mathbf{p}_2) = -(\mathbf{p}_3 - \mathbf{p}_2) + \mathbf{p}_5 - \mathbf{p}_4$$
$$(\mathbf{p}_1 - \mathbf{p}_2) + 2(\mathbf{p}_3 - \mathbf{p}_2) + \mathbf{p}_4 = \mathbf{p}_5$$

Finally, again from C^1 continuity, $\mathbf{p}_4 = \mathbf{p}_3 + \mathbf{p}_3 - \mathbf{p}_2$. Thus,

$\mathbf{p}_5 = \mathbf{p}_3 + (\mathbf{p}_1 - \mathbf{p}_2) + 3(\mathbf{p}_3 - \mathbf{p}_2)$

So $\mathbf{p}_4$ and $\mathbf{p}_5$ are uniquely determined by $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$. Only $\mathbf{p}_6$ can be chosen arbitrarily.

8. From Exercise 4(b), $\mathbf{x}''(0) = \mathbf{p}_0 - 2\mathbf{p}_1 + \mathbf{p}_2$ and $\mathbf{x}''(1) = \mathbf{p}_1 - 2\mathbf{p}_2 + \mathbf{p}_3$. Use the formula for $\mathbf{x}''(0)$, with the shifted control points for $\mathbf{y}(t)$, to get

$$\mathbf{y}''(0) = \mathbf{p}_1 - 2\mathbf{p}_2 + 2\mathbf{p}_3 = \mathbf{x}''(1)$$

Thus the curve has C^2 continuity at $\mathbf{x}(1)$.

9. Write a vector of the polynomial weights for $\mathbf{x}(t)$, expand the polynomial weights and factor the vector as $M_B\mathbf{u}(t)$:

$$\begin{bmatrix} 1-4t+6t^2-4t^3+t^4 \\ 4t-12t^2+12t^3-4t^4 \\ 6t^2-12t^3+6t^4 \\ 4t^3-4t^4 \\ t^4 \end{bmatrix} = \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ 0 & 4 & -12 & 12 & -4 \\ 0 & 0 & 6 & -12 & 6 \\ 0 & 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \\ t^4 \end{bmatrix}, \quad M_B = \begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ 0 & 4 & -12 & 12 & -4 \\ 0 & 0 & 6 & -12 & 6 \\ 0 & 0 & 0 & 4 & -4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

10. Write a vector of the polynomial weights for $\mathbf{x}(t)$, expand the polynomial weights, taking care to write the terms in ascending powers of t, and factor the vector as $M_S\mathbf{u}(t)$:

$$\frac{1}{6}\begin{bmatrix} 1-3t+3t^2-t^3 \\ 4-6t^2+3t^3 \\ 1+3t+3t^2-3t^3 \\ t^3 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 1 & -3 & 3 & -1 \\ 4 & 0 & -6 & 3 \\ 1 & 3 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix} = M_S\mathbf{u}(t), \quad M_S = \frac{1}{6}\begin{bmatrix} 1 & -3 & 3 & -1 \\ 4 & 0 & -6 & 3 \\ 1 & 3 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

11. True. See equation (2).

12. False. The essential properties are preserved under translations as well as linear transformations. See the comment after Figure 1.

13. False. Example 1 shows that the tangent vector $\mathbf{x}'(t)$ at $\mathbf{p}_0$ is <u>two</u> times the directed line segment from $\mathbf{p}_0$ to $\mathbf{p}_1$.

14. True. This is the definition of G^0 continuity at a point.

15. True. See Example 2.

16. False. The Bézier basis matrix is a matrix of the polynomial coefficients of the control points. See the definition before equation (4).

17. **a**. From (12), $\mathbf{q}_1 - \mathbf{q}_0 = \frac{1}{2}(\mathbf{p}_1 - \mathbf{p}_0) = \frac{1}{2}\mathbf{p}_1 - \frac{1}{2}\mathbf{p}_0$. Since $\mathbf{q}_0 = \mathbf{p}_0$, $\mathbf{q}_1 = \frac{1}{2}(\mathbf{p}_1 + \mathbf{p}_0)$.

b. From (13), $8(\mathbf{q}_3 - \mathbf{q}_2) = -\mathbf{p}_0 - \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3$. So $8\mathbf{q}_3 + \mathbf{p}_0 + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3 = 8\mathbf{q}_2$.

c. Use (8) to substitute for $8\mathbf{q}_3$, and obtain

$8\mathbf{q}_2 = (\mathbf{p}_0 + 3\mathbf{p}_1 + 3\mathbf{p}_2 + \mathbf{p}_3) + \mathbf{p}_0 + \mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3 = 2\mathbf{p}_0 + 4\mathbf{p}_1 + 2\mathbf{p}_2$

Then divide by 8, regroup the terms, and use part (a) to obtain

$$\mathbf{q}_2 = \tfrac{1}{4}\mathbf{p}_0 + \tfrac{1}{2}\mathbf{p}_1 + \tfrac{1}{4}\mathbf{p}_2 = (\tfrac{1}{4}\mathbf{p}_0 + \tfrac{1}{4}\mathbf{p}_1) + (\tfrac{1}{4}\mathbf{p}_1 + \tfrac{1}{4}\mathbf{p}_2) = \tfrac{1}{2}\mathbf{q}_1 + \tfrac{1}{4}(\mathbf{p}_1 + \mathbf{p}_2)$$
$$= \tfrac{1}{2}(\mathbf{q}_1 + \tfrac{1}{2}(\mathbf{p}_1 + \mathbf{p}_2))$$

18. a. $3(\mathbf{r}_3 - \mathbf{r}_2) = \mathbf{z}'(1)$, by (9) with $\mathbf{z}'(1)$ and $\mathbf{r}_j$ in place of $\mathbf{x}'(1)$ and $\mathbf{p}_j$.

$\mathbf{z}'(1) = .5\mathbf{x}'(1)$, by (11) with $t = 1$.

$.5\mathbf{x}'(1) = (.5)3(\mathbf{p}_3 - \mathbf{p}_2)$, by (9).

b. From part (a), $6(\mathbf{r}_3 - \mathbf{r}_2) = 3(\mathbf{p}_3 - \mathbf{p}_2)$, $\mathbf{r}_3 - \mathbf{r}_2 = \frac{1}{2}\mathbf{p}_3 - \frac{1}{2}\mathbf{p}_2$, and $\mathbf{r}_3 - \frac{1}{2}\mathbf{p}_3 + \frac{1}{2}\mathbf{p}_2 = \mathbf{r}_2$. Since $\mathbf{r}_3 = \mathbf{p}_3$, this equation becomes $\mathbf{r}_2 = \frac{1}{2}(\mathbf{p}_3 + \mathbf{p}_2)$.

c. $3(\mathbf{r}_1 - \mathbf{r}_0) = \mathbf{z}'(0)$, by (9) with $\mathbf{z}'(0)$ and $\mathbf{r}_j$ in place of $\mathbf{x}'(0)$ and $\mathbf{p}_j$.

$\mathbf{z}'(0) = .5\mathbf{x}'(.5)$, by (11) with $t = 0$.

d. Part (c) and (10) show that $3(\mathbf{r}_1 - \mathbf{r}_0) = \frac{3}{8}(-\mathbf{p}_0 - \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3)$. Multiply by $\frac{8}{3}$ and rearrange to

obtain $8\mathbf{r}_1 = -\mathbf{p}_0 - \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + 8\mathbf{r}_0$.

e. From (8), $8\mathbf{r}_0 = \mathbf{p}_0 + 3\mathbf{p}_1 + 3\mathbf{p}_2 + \mathbf{p}_3$. Substitute into the equation from part (d), and obtain $8\mathbf{r}_1 = 2\mathbf{p}_1 + 4\mathbf{p}_2 + 2\mathbf{p}_3$. Divide by 8 and use part (b) to obtain

$$\mathbf{r}_1 = \tfrac{1}{4}\mathbf{p}_1 + \tfrac{1}{2}\mathbf{p}_2 + \tfrac{1}{4}\mathbf{p}_3 = (\tfrac{1}{4}\mathbf{p}_1 + \tfrac{1}{4}\mathbf{p}_2) + \tfrac{1}{4}(\mathbf{p}_2 + \mathbf{p}_3) = \tfrac{1}{2}\bullet\tfrac{1}{2}(\mathbf{p}_1 + \mathbf{p}_2) + \tfrac{1}{2}\mathbf{r}_2$$

Interchange the terms on the right, and obtain $\mathbf{r}_1 = \frac{1}{2}[\mathbf{r}_2 + \frac{1}{2}(\mathbf{p}_1 + \mathbf{p}_2)]$.

19. a. From (11), $\mathbf{y}'(1) = .5\mathbf{x}'(.5) = \mathbf{z}'(0)$.

b. Observe that $\mathbf{y}'(1) = 3(\mathbf{q}_3 - \mathbf{q}_2)$. This follows from (9), with $\mathbf{y}(t)$ and its control points in

place of $\mathbf{x}(t)$ and its control points. Similarly, for $\mathbf{z}(t)$ and its control points, $\mathbf{z}'(0) = 3(\mathbf{r}_1 - \mathbf{r}_0)$. By part (a) $3(\mathbf{q}_3 - \mathbf{q}_2) = 3(\mathbf{r}_1 - \mathbf{r}_0)$. Replace $\mathbf{r}_0$ by $\mathbf{q}_3$, and obtain $\mathbf{q}_3 - \mathbf{q}_2 = \mathbf{r}_1 - \mathbf{q}_3$, and hence $\mathbf{q}_3 = (\mathbf{q}_2 + \mathbf{r}_1)/2$.

c. Set $\mathbf{q}_0 = \mathbf{p}_0$ and $\mathbf{r}_3 = \mathbf{p}_3$. Compute $\mathbf{q}_1 = (\mathbf{p}_0 + \mathbf{p}_1)/2$ and $\mathbf{r}_2 = (\mathbf{p}_2 + \mathbf{p}_3)/2$. Compute $\mathbf{m} = (\mathbf{p}_1 + \mathbf{p}_2)/2$.
Compute $\mathbf{q}_2 = (\mathbf{q}_1 + \mathbf{m})/2$ and $\mathbf{r}_1 = (\mathbf{m} + \mathbf{r}_2)/2$. Compute $\mathbf{q}_3 = (\mathbf{q}_2 + \mathbf{r}_1)/2$ and set $\mathbf{r}_0 = \mathbf{q}_3$.

20. A Bézier curve is completely determined by its four control points. Two are given directly: $\mathbf{p}_0 = \mathbf{x}(0)$ and $\mathbf{p}_3 = \mathbf{x}(1)$. From equation (9), $\mathbf{x}'(0) = 3(\mathbf{p}_1 - \mathbf{p}_0)$ and $\mathbf{x}'(1) = 3(\mathbf{p}_3 - \mathbf{p}_2)$. Solving gives

$\mathbf{p}_1 = \mathbf{p}_0 + \frac{1}{3}\mathbf{x}'(0)$ and $\mathbf{p}_2 = \mathbf{p}_3 - \frac{1}{3}\mathbf{x}'(1)$.

21. a. The quadratic curve is $\mathbf{w}(t) = (1-t)^2\mathbf{p}_0 + 2t(1-t)\mathbf{p}_1 + t^2\mathbf{p}_2$. From Example 1, the tangent vectors at the endpoints are $\mathbf{w}'(0) = 2\mathbf{p}_1 - 2\mathbf{p}_0$ and $\mathbf{w}'(1) = 2\mathbf{p}_2 - 2\mathbf{p}_1$. Denote the control points of $\mathbf{x}(t)$ by $\mathbf{r}_0$, $\mathbf{r}_1$, $\mathbf{r}_2$, and $\mathbf{r}_3$. Then
$\mathbf{r}_0 = \mathbf{x}(0) = \mathbf{w}(0) = \mathbf{p}_0$ and $\mathbf{r}_3 = \mathbf{x}(1) = \mathbf{w}(1) = \mathbf{p}_2$
From equation (9) or Exercise 3(a) (using $\mathbf{r}_i$ in place of $\mathbf{p}_i$) and Example 1,

$-3\mathbf{r}_0 + 3\mathbf{r}_1 = \mathbf{x}'(0) = \mathbf{w}'(0) = 2\mathbf{p}_1 - 2\mathbf{p}_0$

So $-\mathbf{p}_0 + \mathbf{r}_1 = \dfrac{2\mathbf{p}_1 - 2\mathbf{p}_0}{3}$ and

$$\mathbf{r}_1 = \frac{\mathbf{p}_0 + 2\mathbf{p}_1}{3} \qquad (i)$$

Similarly, using the tangent data at $t = 1$, along with equation (9) and Example 1, yields

$-3\mathbf{r}_2 + 3\mathbf{r}_3 = \mathbf{x}'(1) = \mathbf{w}'(1) = 2\mathbf{p}_2 - 2\mathbf{p}_1$,

$-\mathbf{r}_2 + \mathbf{p}_2 = \dfrac{2\mathbf{p}_2 - 2\mathbf{p}_1}{3}$, $\mathbf{r}_2 = \mathbf{p}_2 - \dfrac{2\mathbf{p}_2 - 2\mathbf{p}_1}{3}$, and

$$\mathbf{r}_2 = \frac{2\mathbf{p}_1 + \mathbf{p}_2}{3} \qquad (ii)$$

b. Write the standard formula (7) in this section, with $\mathbf{r}_i$ in place of $\mathbf{p}_i$ for $i = 1, \ldots, 4$, and then replace $\mathbf{r}_0$ and $\mathbf{r}_3$ by $\mathbf{p}_0$ and $\mathbf{p}_2$, respectively:

$$\mathbf{x}(t) = (1 - 3t + 3t^2 - t^3)\mathbf{p}_0 + (3t - 6t^2 + 3t^3)\mathbf{r}_1 + (3t^2 - 3t^3)\mathbf{r}_2 + t^3\mathbf{p}_2 \qquad (iii)$$

Use the formulas (i) and (ii) for $\mathbf{r}_1$ and $\mathbf{r}_2$ to examine the second and third terms in (iii):

$$(3t - 6t^2 + 3t^3)\mathbf{r}_1 = \tfrac{1}{3}(3t - 6t^2 + 3t^3)\mathbf{p}_0 + \tfrac{2}{3}(3t - 6t^2 + 3t^3)\mathbf{p}_1$$

$$= (t - 2t^2 + t^3)\mathbf{p}_0 + (2t - 4t^2 + 2t^3)\mathbf{p}_1$$

$$(3t^2 - 3t^3)\mathbf{r}_2 = \tfrac{2}{3}(3t^2 - 3t^3)\mathbf{p}_1 + \tfrac{1}{3}(3t^2 - 3t^3)\mathbf{p}_2$$

$$= (2t^2 - 2t^3)\mathbf{p}_1 + (t^2 - t^3)\mathbf{p}_2$$

When these two results are substituted in (iii), the coefficient of $\mathbf{p}_0$ is

$(1 - 3t + 3t^2 - t^3) + (t - 2t^2 + t^3) = 1 - 2t + t^2 = (1 - t)^2$

The coefficient of $\mathbf{p}_1$ is

$(2t - 4t^2 + 2t^3) + (2t^2 - 2t^3) = 2t - 2t^2 = 2t(1 - t)$

The coefficient of $\mathbf{p}_2$ is $(t^2 - t^3) + t^3 = t^2$. So $\mathbf{x}(t) = (1 - t)^2\mathbf{p}_0 + 2t(1 - t)\mathbf{p}_1 + t^2\mathbf{p}_2$, which shows that $\mathbf{x}(t)$ is the quadratic Bézier curve $\mathbf{w}(t)$.

22. $\begin{bmatrix} \mathbf{p}_0 \\ -3\mathbf{p}_0 + 3\mathbf{p}_1 \\ 3\mathbf{p}_0 - 6\mathbf{p}_1 + 3\mathbf{p}_2 \\ -\mathbf{p}_0 + 3\mathbf{p}_1 - 3\mathbf{p}_2 + \mathbf{p}_3 \end{bmatrix}$

Chapter 8 - Supplementary Exercises

1. True. This is the definition.

2. True. See Equation (1) in Section 8.1 with the roles of $\mathbf{v}_1$ and $\mathbf{v}_2$ reversed.

3. False. A hyperplane is a 4-dimensional flat only if it is a subset of $\mathbb{R}^5$.

4. False. Two lines in $\mathbb{R}^3$ can have an empty intersection without being translates of each other.

5. True. If you translate a subspace by a point in the subspace, the translated set coincides with the original subspace. So every subspace is a translate of a subspace. Hence it is a flat.

6. True. A subspace S contains all linear combinations of points in S, so it contains all affine combinations of points in S. Theorem 2 says that S is affine. Or, combine the previous T/F exercise with Theorem 3 which says that every flat is affine.

7. True. See the comment after the definition of affine dependence.

8. False. See Example 1 in Section 8.2. The set $\{\mathbf{p}, \mathbf{q}, \mathbf{s}\}$ is affinely independent, but those three vectors in $\mathbb{R}^2$ must be linearly dependent. In general, three non-collinear points in $\mathbb{R}^2$ will be affinely independent, but linearly dependent.

9. False. See the comment prior to Figure 5 in Section 8.2.

10. False. This is only true for points $\mathbf{p}$ in aff S (or if $k = n + 1$).

11. True. This is the definition.

12. True. See the comment prior to Example 1 in Section 8.3.

13. False. The scalar t must satisfy $0 \le t \le 1$.

14. True. Given points $\mathbf{p}$, $\mathbf{q}$ in an affine set S, we have $(1 - t)\mathbf{p} + t\mathbf{q} \in S$ for all t in $\mathbb{R}$. It follows that $(1 - t)\mathbf{p} + t\mathbf{q} \in S$ for all t such that $0 \le t \le 1$, and S is convex.

15. True. In $\mathbb{R}^2$, both a line and a hyperplane have dimension one.

16. True. In $\mathbb{R}^1$, the dimension of a hyperplane is zero and the dimension of a line is one.

17. True. Since a compact set is closed, this result follows from Theorem 12 in Section 8.4.

18. True. The three edges of a triangle consist of infinitely many points, and their convex hull is the triangle.

19. True. See the comment prior to Theorem 15 in Section 8.5.

20. False. The first part is true, but in general the curve does not even go through $\mathbf{p}_1$. $\mathbf{w}'(1)$ has the same direction as the tangent to the curve at the terminal point, $\mathbf{p}_2$. See Example 1 in Section 8.6.

21. True. See the comment following Figure 2 in Section 8.6.

22. If $k > n + 1$, then $k - 1 > n$, and the $k - 1$ vectors in $\{\mathbf{v}_2 - \mathbf{v}_1, \ldots, \mathbf{v}_k - \mathbf{v}_1\}$ would be linearly independent by Theorem 5 in Section 8.2. But any set containing more than n vectors in $\mathbb{R}^n$ must be linearly dependent (Theorem 10 in Section 4.5). This contradiction implies $k \le n + 1$.

23. Let $\mathbf{y} \in F$. Then $U = F - \mathbf{y}$ and $V = G - \mathbf{y}$ are k-dimensional subspaces with $U \subseteq V$. Let $B = \{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ be a basis for U. Since dim $V = k$, B is also a basis for V. Hence $U = V$, and $F = U + \mathbf{y} = V + \mathbf{y} = G$.

24. This statement is true. Given any $\mathbf{p}$, $\mathbf{q}$ in S, let T be the set of points of the form $(1 - t)\mathbf{p} + t\mathbf{q}$, where $0 < t < 1$. This is the line segment from $\mathbf{p}$ to $\mathbf{q}$ without the endpoints $\mathbf{p}$ and $\mathbf{q}$. Since we know that $\mathbf{p}$ and $\mathbf{q}$ are in S, it follows that $T \subseteq S$ if and only if $\overline{\mathbf{pq}} \subseteq S$ if and only if S is convex.

25. Suppose $F_1 \cap F_2 \neq \varnothing$. Then there exist $\mathbf{v}_1$ and $\mathbf{v}_2$ in V such that $\mathbf{x}_1 + \mathbf{v}_1 = \mathbf{x}_2 + \mathbf{v}_2$. That is, $\mathbf{x}_1 = \mathbf{x}_2 + \mathbf{v}_2 - \mathbf{v}_1$ and $\mathbf{x}_2 = \mathbf{x}_1 + \mathbf{v}_1 - \mathbf{v}_2$.Then for all $\mathbf{v}$ in V we have $\mathbf{x}_1 + \mathbf{v} = \mathbf{x}_2 + (\mathbf{v}_2 - \mathbf{v}_1 + \mathbf{v}) \in \mathbf{x}_2 + V$ since V is a subspace. Thus $\mathbf{x}_1 + V \subseteq \mathbf{x}_2 + V$. Likewise, $\mathbf{x}_2 + \mathbf{v} = \mathbf{x}_1 + (\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}) \in \mathbf{x}_1 + V$, so $\mathbf{x}_2 + V \subseteq \mathbf{x}_1 + V$. Hence, $F_1 = F_2$.

26. If $\boldsymbol{x} \in H_1$, then there exists $\mathbf{v}$ in H such that $\mathbf{x} = \mathbf{v} + 3\mathbf{p}$. Since f is linear we have $f(\mathbf{x}) = f(\mathbf{v} + 3\mathbf{p}) = f(\mathbf{v}) + 3f(\mathbf{p}) = 7 + (3)(2) = 13$, so $d = 13$.

27. Let $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}$ be a basis for V. Since $\mathbf{p} \notin V$, the n vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}, \mathbf{p}$ are linearly independent and hence a basis for $\mathbb{R}^n$. Thus any $\mathbf{x}$ in $\mathbb{R}^n$ has a unique representation as
$\mathbf{x} = c_1\mathbf{v}_1 + \cdots + c_{n-1}\mathbf{v}_{n-1} + c\mathbf{p} = \mathbf{v} + c\mathbf{p}$, where $\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_{n-1}\mathbf{v}_{n-1} \in V$.

28. If $\mathbf{x} \in \overline{\mathbf{pq}}$, then $\mathbf{x} = (1 - t)\mathbf{p} + t\mathbf{q}$ for some t such that $0 \le t \le 1$. Since f is linear we have $f(\mathbf{x}) = f\big((1 - t)\mathbf{p} + t\mathbf{q}\big) = (1 - t)f(\mathbf{p}) + tf(\mathbf{q}) = (1 - t)m + tm = m$.

29. Suppose $\mathbf{x} \in \lambda B(\mathbf{p}, \delta)$. Then there exists $\mathbf{y} \in B(\mathbf{p}, \delta)$ such that $\mathbf{x} = \lambda\mathbf{y}$. Since $\mathbf{y} \in B(\mathbf{p}, \delta)$, we have $\|\mathbf{y} - \mathbf{p}\| < \delta$, $\|\lambda\mathbf{y} - \lambda\mathbf{p}\| < \lambda\delta$, and $\|\mathbf{x} - \lambda\mathbf{p}\| < \lambda\delta$, so $\mathbf{x} \in B(\lambda\mathbf{p}, \lambda\delta)$. Conversely, suppose $\mathbf{z} \in B(\lambda\mathbf{p}, \lambda\delta)$. Then $\|\mathbf{z} - \lambda\mathbf{p}\| < \lambda\delta$, $\|(1/\lambda)\mathbf{z} - \mathbf{p}\| < \delta$, and $(1/\lambda)\mathbf{z} \in B(\mathbf{p}, \delta)$. Thus there exists $\mathbf{w} \in B(\mathbf{p}, \delta)$ such that $(1/\lambda)\mathbf{z} = \mathbf{w}$. That is, $\mathbf{z} = \lambda\mathbf{w}$ and $\mathbf{z} \in \lambda B(\mathbf{p}, \delta)$.

30. **a.** $\mathbf{v}_2 - \mathbf{v}_1 = (1, 0, 0, 1)$, $\mathbf{v}_3 - \mathbf{v}_1 = (0, 1, 0,1)$, and $\mathbf{v}_4 - \mathbf{v}_1 = (0, 1, 1, 2)$. These are easily shown to be linearly independent.

b. The answer to part (a) shows that A is a translate of a 3-dimensional subspace, so dim $A = 3$. Likewise, dim $B = 3$ since it is a hyperplane in $\mathbb{R}^4$. It is easy to check that $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ are all in B. Thus aff $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} \subseteq B$ since B is affine (Theorem 2). That means A and B are 3-dimensional flats with $A \subseteq B$. So $A = B$ by Exercise 23.

31. The positive hull of S is a cone with vertex (0, 0) containing the positive y axis and with sides on the lines $y = \pm x$.

32. We have $\mathbf{p} = \mathbf{v}_1 + \mathbf{v}_2 + 2\mathbf{v}_3$, a positive combination. Also, $\mathbf{p} = -2\mathbf{v}_1 + 4\mathbf{v}_2 - \mathbf{v}_3$, an affine combination. But $\mathbf{p}$ is clearly outside the convex hull of S, a triangle with vertices $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

33. The set in Exercise 31 consists of exactly two vectors, say $\mathbf{u}_1$ and $\mathbf{u}_2$. These vectors form a basis for $\mathbb{R}^2$. Thus any other point $\mathbf{p}$ in $\mathbb{R}^2$ has a *unique* representation as a linear combination of $\mathbf{u}_1$ and $\mathbf{u}_2$. In Exercise 32, there were two different linear combinations giving $\mathbf{p}$, one affine and one positive, but no single linear combination with both properties.

34. Let $\mathbf{x} \in$ pos (conv S). Then there exist points $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in conv S such that $\mathbf{x} = c_1\mathbf{v}_1 + \cdots c_k\mathbf{v}_k$, and all $c_i \ge 0$. But each $\mathbf{v}_i$ is a convex combination of $n + 1$ or fewer points of S (by Caratheodory's

Theorem). That is, there exist $\mathbf{v}_{i,j}$ in S and $d_{i,j} \geq 0$ with $d_{i,1} + \cdots + d_{i,n+1} = 1$ for all i such that $\mathbf{v}_i = d_{i,1}\mathbf{v}_{i,1} + \cdots + d_{i,n+1}\mathbf{v}_{i,n+1}$. Now we have

$\mathbf{x} = c_1(d_{1,1}\mathbf{v}_{1,1} + \cdots + d_{1,n+1}\mathbf{v}_{1,n+1}) + \cdots + c_k(d_{k,1}\mathbf{v}_{k,1} + \cdots + d_{k,n+1}\mathbf{v}_{k,n+1})$, so $\mathbf{x} \in$ pos S.

The converse is immediate, since $S \subseteq$ conv S.

35. Suppose $\mathbf{x} \in$ pos S. Then $\mathbf{x} = c_1\mathbf{v}_1 + \cdots c_k\mathbf{v}_k$, where $\mathbf{v}_i \in S$ and all $c_i \geq 0$. Let $d = \sum_{i=1}^{k} c_i$. If $d = 0$, then all $c_i = 0$, so $\mathbf{x} = \mathbf{0}$ and $\mathbf{x} = \lambda\mathbf{s}$ for $\lambda = 0$ and any $\mathbf{s}$ in S. If $d \neq 0$, then $\frac{1}{d}\mathbf{x} = \frac{c_1}{d}\mathbf{v}_1 + \cdots + \frac{c_k}{d}\mathbf{v}_k \in S$ since S is convex. Thus $\frac{1}{d}\mathbf{x} = \mathbf{s}$ for some $\mathbf{s}$ in S, and $\mathbf{x} = \lambda\mathbf{s}$.

Conversely, if $\mathbf{x} = \lambda\mathbf{s}$ for some $\lambda \geq 0$ and some $\mathbf{x}$ in S, then $\mathbf{x}$ is a positive combination of a point in S and $\mathbf{x} \in$ pos S.

36. See Figure 1.

37. Algebraically, they all have one coefficient equal to zero. Geometrically, they all lie on one of the lines extending the sides of the triangle.

38. Algebraically, they all have no zero coefficients and exactly one negative coefficient. Geometrically, they all lie in a region bounded by one side of triangle **abc**.

39. Algebraically, they all have no zero coefficients and exactly one negative coefficient. Geometrically, they all lie in a region bounded by one side of triangle **abc**. Their **b** coefficient is negative and they are in the region bounded by side **ac**, instead of having their **a** coefficient negative and being in the region bounded by side **bc**.

40. Algebraically, they all have no zero coefficients and exactly two negative coefficients. Geometrically, they all lie in a region beyond a vertex of triangle **abc**.

41. See Figure 2. The coefficients of **c** decrease as you move from the lower left to the upper right. Points $\mathbf{p}_3$, $\mathbf{p}_4$ and $\mathbf{p}_{12}$ have the same **c** coefficient since the line through them is parallel to the line through **a** and **b**.

42. Since $2\mathbf{a} + 2\mathbf{b} + 3\mathbf{c} = \mathbf{0}$, we can divide by 7 to get $\frac{2}{7}\mathbf{a} + \frac{2}{7}\mathbf{b} + \frac{3}{7}\mathbf{c} = \mathbf{0}$. Algebraically, it is the only point where all the coefficients are positive. Geometrically, it is the only point inside triangle **abc**.

43. The sum of the "**abc**" coefficients in (1, 1) is 3. So if we subtract 2 times the "**abc**" coefficients in **0**, the new coefficients will sum to 1:

$$(1, 1) = \mathbf{a} + \mathbf{b} + \mathbf{c} - 2\left(\tfrac{2}{7}\mathbf{a} + \tfrac{2}{7}\mathbf{b} + \tfrac{3}{7}\mathbf{c}\right) = \tfrac{3}{7}\mathbf{a} + \tfrac{3}{7}\mathbf{b} + \tfrac{1}{7}\mathbf{c}.$$

Once again the coefficients are all positive and the point (1, 1) is inside triangle **abc**.

44. There is no region corresponding to all three coefficients being negative. If the sum of three numbers is equal to one, at least one of the numbers must be positive.

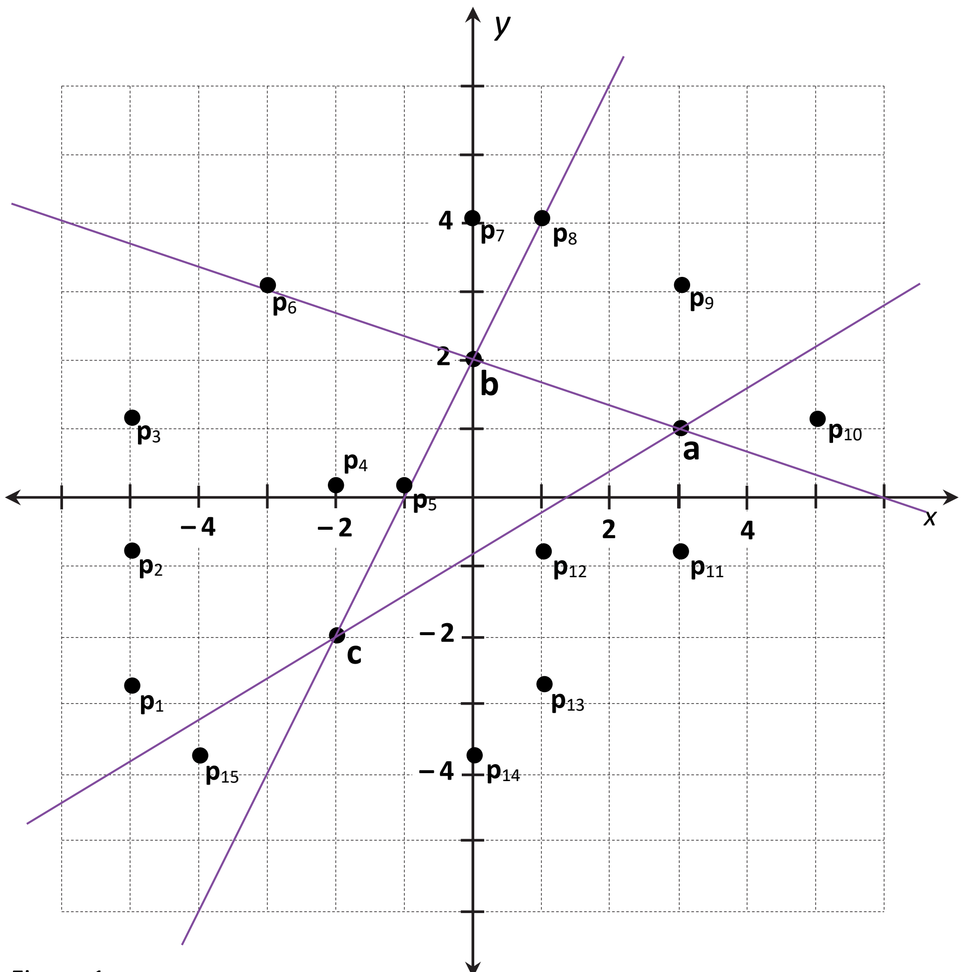


Figure 1

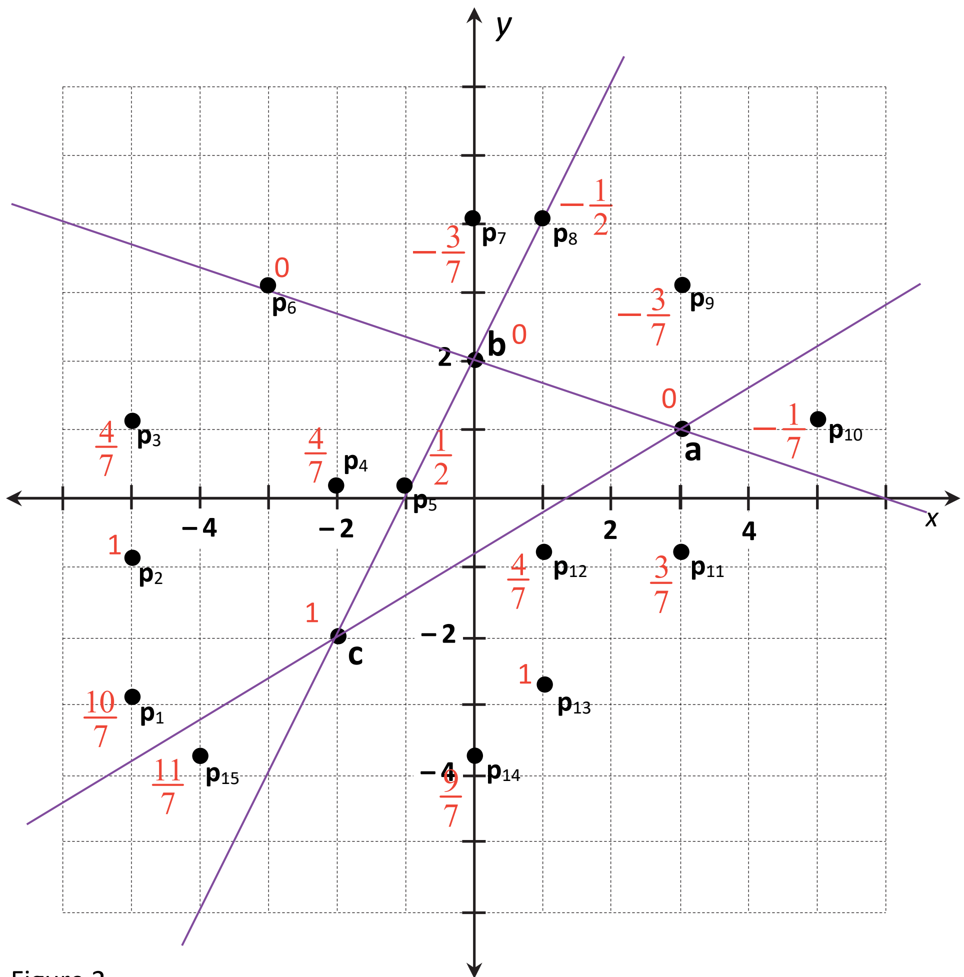

y
x
2
−2
−4
−4
−2
2
4
a
b
c
p1
p2
p3
p4
p5
p6
p7
p8
p9
p10
p11
p12
p13
p14
p15
0
0
0
1
1
1
1
−1/2
−3/7
−3/7
−1/7
4/7
4/7
1/2
4/7
3/7
10/7
11/7
9/7

Figure 2